INTEGRATION IN LOCALLY CONVEX SPACES

By

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In this dissertation, we define an integral of a B-valued function with respect to an $L(B,X)$-valued measure, where B is a Banach space, X is a locally convex space and $L(B,X)$ is the space of linear, continuous operators from B to X. This integral is an extension of the integral defined by D.R.Lewis in which B is the complex number field C.

An extensive development of Lewis' integration theory has been done by I.Kluvanek and G.Knowles. They study the Lebesgue space and prove many results on it, including the criteria for quasi-completeness. However, compared to the theory of Brooks and Dinculeanu on bilinear integration in Banach spaces, weakly sequential completeness and weakly sequential compactness are not discussed. Using the methods motivated by Brooks and Dinculeanu's theory, we obtain sufficient conditions for these two properties of the Lebesgue space $L^1_B(m)$. In particular, when B is reduced to C, we contribute these results to the theory developed by Lewis, Kluvanek and Knowles.

The way we study the weakly sequential completeness is as follows. First, we observe that $L^1_B(m)$ is continuously embedded in a larger space $L^1_B(\Lambda_m)$, where Λ_m is the family of p-Rybakov control measures for the vector measure m. The family Λ_m is obtained by extending a Rybakov-type theorem of Drewnowski. Then, we prove that if m has the Beppo-Levi property and B is reflexive, the weakly sequential completeness of $L^1_B(\Lambda_m)$ implies that of $L^1_B(m)$. In particular, if X is a Fréchet space, then $L^1_B(\Lambda_m)$ is weakly sequentially complete and therefore so is $L^1_B(m)$ if m has the Beppo-Levi property and B is reflexive. On the one hand, if X is a Banach space, this result is parallel to the criterion in Brooks and Dinculeanu's theory. On the other hand, for Lewis' case, it implies that the Lebesgue space is weakly sequentially complete if X contains no copy of c_0, for example, if X is a nuclear space.

Similarly we obtain the result on weakly sequential compactness which is parallel to the one in Brooks and Dinculeanu's theory. We then correct some mistakes in the monograph of Kluvanek and Knowles which are related to the discussions on control systems defined by vector measures. These discussions are also introduced in this dissertation. Especially, three topics are included: the bang-bang principle, time-optimal control, and controllability.

CHAPTER 1
INTRODUCTION

In this dissertation, a new integral in locally convex spaces is defined and the resulting Lebesgue space is studied. Our integral extends the integral with vector measures in locally convex spaces defined by Lewis [19]. Many results in Brooks and Dinculeanu [6] are converted to fit our new setting and then some new results are obtained for the Lewis integration theory.

In order to locate our integral among others in a overall picture, in this chapter we first review some special integrals in different categories. We then introduce the motivation and methodology for this dissertation. Finally, we highlight some of our main results.

1.1 Vector Integration

Integrands and measures are the main components in integration theory. One major development in integration theory is extending the range of the integrands and measures.

Various authors have extended integration theory to Banach spaces (for example, Bochner [4], Birkhoff [3], Gelfand [14], Pettis [22], Price [24], Bartle, Dunford and Schwartz [2], Day [9] and Bartle [1]). As a result the integrals become "vector integrals." There are different vector integrals in Banach spaces. If the integrand is vector valued and the measure is scalar, we have the Bochner integral (see Bochner [4]) which is a generalization of the Lebesgue integral (see Rudin [29]) with absolute value signs replaced by norm signs. Symmetrically, if the integrand is scalar and

the measure is vector valued, we have the integral of Bartle-Dunford-Schwartz [2]. More generally, we have to consider the case where both the integrand and measure are vector valued, called the bilinear case. For the bilinear case, we have the integral of Dinculeanu [11] if the measure is of finite variation and the integral of Brooks-Dinculeanu [6] if the measure is of finite semi-variation.

Integrals in Banach spaces have been extended to integrals in locally convex spaces. In the case when the integrand is locally convex space valued and the measure is scalar, some integrals are defined by Phillips [23] and Richart [26]. Recently a locally convex space integral has been defined by Reinke [25]. If we interchange the values of the integrand and measure, that is, if the integrand is scalar valued and the measure is locally convex space valued, then we have the integral of Lewis [19], which is studied extensively by Kluvanek and Knowles [17]. Although there has not been a general bilinear integral in locally convex spaces, a special one first developed by Brooks is presented in Reinke [25] in which it is assumed that the range space of the integrand is a nuclear dual space. The purpose of the present dissertation is to define another kind of bilinear integral in locally convex spaces.

In this dissertation, the range space of the integrand is denoted by B and the range space of the measure m is denoted by $L(B, X)$, the space of all continuous linear operators from a Banach space B to a locally convex space X with the topology determined by a family P of semi-norms. The integral is defined so that it is an extension of the integral in locally convex spaces defined by Lewis.

After the integral is defined, the resulting Lebesgue space $L^1_B(m)$, the space of classes of integrable functions, is investigated. As a locally convex space, typical questions about $L^1_B(m)$ must be answered. When is a sequence in $L^1_B(m)$ convergent? When is $L^1_B(m)$ complete or weakly sequentially complete? When is a set in $L^1_B(m)$

weakly sequentially compact? The answers to these questions form the core of our integration theory.

1.2 Motivation and Methodology

The motivation to establish the integration theory presented in this dissertation is based on the following considerations. First of all, it is the first step in the attempt of defining a general bilinear integration in locally convex spaces. Secondly, although the integration of Lewis is extensively studied by Kluvanek and Knowles, some topics such as weakly sequential completeness and weakly sequential compactness are yet to be discussed. We know that criteria on these properties are available in Brooks and Dinculeanu [6] for the Banach space case. In order to convert these results to fit the case of Lewis, we shall need a bridge to connect these two integration theories, which turns out to be a method which extends Lewis' integral to bilinear case so that methods of Brooks and Dinculeanu can be used. Then the results in the Brooks-Dinculeanu theory are converted to our case and in turn applied to the Lewis integration theory.

The establishment of our theory is based on the Rybakov-Drewnowski theorem which guarantees the existence of a family Λ_m of finite positive measures associated to the vector measure m. Noting that $L^1_B(m)$ is continuously embedded in $L^1_B(\Lambda_m)$, the space of classes of measurable functions which is λ-integrable for every $\lambda \in \Lambda_m$, we prove with appropriate conditions, that $L^1_B(m)$ inherits many properties from $L^1_B(\Lambda_m)$, for example, completeness and weakly sequential completeness if $L^1_B(\Lambda_m)$ possesses these properties. This embedding method converts the study of $L^1_B(m)$ to the study of $L^1_B(\Lambda_m)$ which is easier to handle since only scalar measures are involved.

In addition to our integration theory, we shall also introduce the work of Kluvanek and Knowles [17] on control systems defined by vector measures. Their work opens

another field of applications for vector measures and integration in locally convex spaces. This application motivates, in turn, research on integration in locally convex spaces. We point out and correct some mistakes contained in the discussion of the integration theory in Kluvanek and Knowles [17]. These corrections allow the results to be applied to control theory.

Three topics on control theory will be introduced, the bang-bang principle, time optimal control and controllability. The bang-bang principle deals with the existence of the control taking extreme values. For instance, if a control system is defined by a closed Liapunov vector measure, then it has the bang-bang control principle, that is, a point in the state space X which is reachable by a control valued in $[0,1]$ is also reachable by a control taking only the two values, 0 and 1. The time optimal control problem is concerned with how a target is reached in the shortest length of time. In time optimal control problems we have to assume that the target is reachable. Obviously we need criteria to guarantee such reachability. These criteria make up the controllability theory.

1.3 Main Results

Let us highlight some of our main results in this dissertation. We shall use the following notations: (T,S) is a measurable space, B is a Banach space, (X,P) is a locally convex space with the topology determined by a family P of semi-norms, $m : S \to L(B,X)$ is a vector measure, where $L(B,X)$ is the space of continuous linear operators from B to X, and $L^1_B(m)$ is the Lebesgue space generated by the m-integrable functions.

1. The Rybakov-Drewnowski theorem. For each $p \in P$, there exists an element $b_p \in B_1$ and an element $x'_p \in U^0_p$ such that $\lim_{\lambda_p(E)\to 0} \| m \|_p (E) = 0$, where B_1 is the closed unit ball in B, $U^0_p \subset X'$ is the polar set for the closed p-unit ball U_p in

X, $\| m \|_p$ is the p-semi-variation of m and $\lambda_p(E) = | < x'_p, m(.)b_p > |(E)$ is a scalar measure defined by

$$< x'_p, m(.)b_p > (E) =< x'_p, m(E)b_p >$$

for $E \in S$.

This result extends a theorem of Drewnowski on refining the Rybakov theorem. In the Drewnowski theorem, X is assumed to be a Banach space.

2. The Vitali theorem. Suppose X is sequentially complete, (f_n) is a sequence from $L^1_B(m)$, $f : T \to B$ is S-measurable and $f_n \to f$ in m. Then $f \in L^1_B(m)$ and $f_n \to f$ in $L^1_B(m)$ if and only if $\lim_{\|m\|_p(E)\to 0} p(m)(f_n 1_E) = 0$ uniformly for $n = 1, 2, \ldots$, where $p(m)(f_n 1_E)$ is the p-upper integral for functions $f_n 1_E$ and 1_E is the indicator function for $E \in S$.

This theorem is new even when $B = C$. In other words, we establish the Vitali theorem for the Lewis integration theory.

As a corollary to the Vitali theorem, we also obtain the Lebesgue dominated theorem.

3. The Riesz theorem. There is an isometric isomorphism $\phi \leftrightarrow (g, \lambda)$ between $L^1_B(m)'_s$ and $G_{B'}(m)_s$ given by

$$< \phi, f >= \int < g, f > d\lambda.$$

The subindex "s" in the theorem stands for the strong topology for the space, defined as follows. The set $G_{B'}(m)$ is defined as a collection of pairs (g, λ) of S-measurable functions $g : T \to B'$ and finite positive measures λ on S, satisfying the condition that there is a positive number c and a finite subset $\{p_i\}_{i\in I} \subset P$ such that

$$\int | < g, f > | d\lambda \leq c \sup_{i\in I} p_i(m)(f) \quad for \quad f \in L^1_B(m).$$

For a bounded set W in $L^1_B(m)$, we define

$$|\phi|'_W = \sup\{| < \phi, f > | : f \in W\}$$

for $\phi \in L^1_B(m)'$ and

$$|(g,\lambda)|_W = \sup\{\int | < g, f > | d\lambda : f \in W\}$$

for $(g,\lambda) \in G_{B'}(m)$.

Let $\mathcal{W}$ be the set of all bounded sets in $L^1_B(m)$. Then $P' = \{|.|'_W : W \in \mathcal{W}\}$ and $P_G = \{|.|_W : W \in \mathcal{W}\}$ are families of semi-norms on $L^1_B(m)'$ and $G_{B'}(m)$ respectively. We then denote $L^1_B(m)'_s = (L^1_B(m), P')$ and $G_{B'}(m)_s = (G_{B'}, P_G)$.

<u>4. The criterion for completeness.</u> If X is sequentially complete, then the completeness of $L^1_B(\Lambda_m)$ implies the completeness of $L^1_B(m)$.

In particular, if X is a Fréchet space (or an (LB) space), $\Lambda_m = \{\lambda_p : p \in P\}$ is countable (Λ_m has only one element λ, respectively) and hence $L^1_B(\Lambda_m)$ is complete. Consequently, $L^1_B(m)$ is complete.

<u>5. The criterion for weakly sequential completeness.</u> Suppose B' has the Radon-Nikodym property and m has the Beppo-Levi property. Then the weakly sequential completeness of $L^1_B(\Lambda_m)$ implies that of $L^1_B(m)$.

In particular, if B is reflexive and X is a Fréchet space or an (LB) space, then $L^1_B(\Lambda_m)$ is weakly sequentially complete. In this case, if m has the Beppo-Levi property, $L^1_B(m)$ is weakly sequentially complete. This is parallel to the result of Brooks and Dinculeanu [6] when X is a Banach space.

On the other hand, if $B = C$ and X is a Fréchet space or an (LB) space which does not contain a copy of c_0, for example, X is a nuclear space, then $L^1(\Lambda_m)$ is weakly sequentially complete and m has the Beppo-Levi property. Therefore $L^1(m)$ is weakly sequentially complete. This is a new result to the Lewis integration theory.

6. The criterion for weakly sequential compactness. Suppose B is a reflexive Banach space and X is a Fréchet space or an (LB) space. If a set K in $L^1_B(m)$ satisfies

(i) K is bounded,

(ii) for each $p \in P$, $\lim_{\|m\|_p(E)\to 0} p(m)(f1_E) = 0$ uniformly for $f \in K$,

then K is conditionally weakly sequentially compact. If in addition, m has the Beppo-Levi property, then K is relatively weakly sequentially compact.

This result, which is parallel to the result of Brooks and Dinculeanu [6] in the case when X is a Banach space, is new in the Lewis integration theory when $B = C$.

CHAPTER 2
LOCALLY CONVEX SPACES

In this chapter we shall define the notion of locally convex spaces and the notions of some special locally convex spaces such as metrizable spaces, (LF) spaces and nuclear spaces. We shall also introduce the equivalence between the B-P(Bessaga-Pelczynski) property of a locally convex space X and the absence of a copy of the space c_0 in X. Note that only a few definitions and results in the general theory will be presented here. The reader should consult Treves [31], Kluvanek and Knowles [17], Cristescu [8] and Yosida [32] for the pertinent definitions and results.

Definition 1. A nonnegative function p on a vector space X is called a semi-norm if it satisfies the following conditions:

(i) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$;

(ii) $p(cx) = |c|p(x)$ for all $x \in X$ and all $c \in C$, where C is the field of complex numbers.

If in addition, p satisfies the condition

$$x \in X,\ p(x) = 0 \ \ imply\ x = 0,$$

then it is called a norm.

Definition 2. A topological vector space X is said to be a locally convex space if there is a basis of neighborhoods of zero in X consisting of convex sets.

Remark 3. It is known (see Treves [31]) that a topological vector space X is a locally convex space if and only if its topology is determined by a family P of semi-norms on X in the following sense:

(i) each $p \in P$ is continuous on X;

(ii) $\{x \in X : \ sup_{i \in I} p_i(x) < \epsilon\}$ forms a basis of neighborhoods of zero when $\{p_i : \ i \in I\}$ ranges over all the finite subsets of P and ϵ ranges over all positive numbers.

Later on we simply say that a locally convex space X with semi-norms P, denoted by (X, P), to mean that P determines the topology of X in the sense above.

Remark 4. It is clear that if X is a locally convex space with semi-norms P, then a net (x_α) from X converges to an point $x \in X$ if and only if $\lim p(x_\alpha - x) = 0$ for each $p \in P$.

A set $A \subset X$ is bounded if and only if $\{p(x) : x \in A\}$ is bounded for every $p \in P$.

In order to guarantee that every convergent net in X has only one limit, we need the Hausdorff property on X.

Definition 5. A topological space X is said to be Hausdorff if, given any two distinct points x and y of X, there is a neighborhood U of x and a neighborhood V of y which do not intersect, that is, $U \cap V$ is empty.

Remark 6. Suppose X is a locally convex space with semi-norms P. Then the quotient space $X/(\cap_{p \in P} Ker\ p)$ is a locally convex Hausdorff space with semi-norms $\overline{p}$ defined by

$$\overline{p}(\overline{x}) = p(x),$$

where $x \mapsto \overline{x}$ is the canonical map of X onto the quotient space, $Ker\ p = \{x \in X : p(x) = 0\}$, and $p \in P$.

Definition 7. Let X be a topological vector space over C, a functional ϕ on X is a map of X into C. The set X' of all continuous linear functionals on X is called the dual of X.

Remark 8. For a locally convex space X with semi-norms P, $\phi \in X'$ if and only if there exists a constant $c > 0$ and a finite subset $\{p_i : i \in I\} \subset P$ such that

$$|\phi(x)| \leq c \sup_{i \in I} p_i(x)$$

for all $x \in X$.

Remark 9. There are different topologies we can put on X'. The main topology will be the strong topology defined as the topology such that a net in X' converges to zero if and only if it converges to zero uniformly on every bounded subset of X. With this topology X' is a locally convex Hausdorff space.

Definition 10. Let A be a subset of a topological vector space X. The subset of X',

$$\{x' \in X' : \sup_{x \in A} | < x', x > | \leq 1\}$$

is called the polar of A and denoted by A^0, where $< x', x >$ is the value of x' at x.

Remark 11. The family of polars of all bounded subsets of X forms a basis of neighborhoods of zero for the strong topology of X'.

Definition 12. A topological vector space X is

(i)(sequentially) complete if every Cauchy (sequence) net in X converges to a point in X;

(ii)quasi-complete if every closed bounded subspace of X is complete;

(iii)weakly sequentially complete if every weak Cauchy sequence in X weakly converges to a point in X, that is, if (x_n) is a sequence in X with the property that $(< x', x_n >)$ is Cauchy for every $x' \in X'$, then there exists a point $x \in X$ such that

$$\lim_{n \to \infty} < x', x_n > = < x', x >$$

for all $x' \in X'$.

Definition 13. A set K in a topological vector space X is

(i) (sequentially) compact if every (sequence) net from K contains a (subsequence) subnet convergent to a point in K;

(ii) (sequentially) relatively compact if every (sequence) net from K contains a (subsequence)subnet convergent to a point in X;

(iii) (sequentially) conditionally compact if every (sequence) net from K contains a Cauchy (subsequence) subnet; Similarly we define notions of weak (sequential) compactness, weakly (sequentially) relative compactness and weakly (sequentially) conditional compactness.

Definition 14. Let X be a locally convex Hausdorff space. If there is a countable family P of semi-norms such that it determines the topology of X, then X is called a metrizable space. A complete metrizable space is called a Fréchet space.

Remark 15. In a metrizable space, completeness and sequential completeness, compactness and sequential compactness (including weak, relative and conditional compactness) are equivalent.

Definition 16. Let X be the union of an increasing sequence of subspaces $X_n, n = 1, 2, \ldots$. Suppose each X_n is a Fréchet space such that the natural injection of X_n into X_{n+1} is an isomorphism, which means that the topology induced by X_{n+1} on X_n is identical to the topology initially given on X_n. We define a Hausdorff locally convex topology on X as follows: a convex subset V of X is a neighborhood of zero if and only if $V \cap X_n$ is a neighborhood of zero in the Fréchet space X_n for every $n = 1, 2, \ldots$. With this topology, X is called an (LF) space or, equivalently, a countable strict inductive limit of Fréchet spaces. In particular, if X_n is a Banach space for all $n = 1, 2, \ldots$, then X is called an (LB) space or, a countable strict inductive limit of Banach spaces.

Remark 17. Each X_n is a closed subspace of X.

In fact, the notion of an (LF) space is a special case of the following definition.

Definition 18. Let X be a linear vector space and let (X_α) be a family of locally convex spaces. Suppose for each index α, we are given a linear map $\phi_\alpha : X_\alpha \to X$ such that $X = \bigcup_\alpha \phi_\alpha(X_\alpha)$. We then define on X the finest locally convex topology such that all the mappings ϕ_α be continuous. A convex subset U of X is a neighborhood of zero in this topology if, for every α, $U \bigcap \phi_\alpha(X_\alpha)$ is of the form $\phi_\alpha(U_\alpha)$, where U_α is a neighborhood of zero in X_α. When X is equipped with this topology, it is called the inductive limit of the spaces (X_α).

Another special case of the above definition is the notion of direct sum of locally convex spaces.

Definition 19. Suppose X, X_α and ϕ_α are given as in Definition 18 except $X = \cup_\alpha \phi_\alpha(X_\alpha)$ is replaced by the hypothesis that X is the algebraic direct sum of the vector spaces X_α : every element x of X can be written uniquely as a sum $x = \sum_\alpha \phi_\alpha(x_\alpha)$ in which all x_α are equal to zero except possibly a finite number of them. In addition, we assume that every ϕ_α is injective. Then the direct sum topology on X is the finest locally convex topology such that all the mappings ϕ_α are continuous. We say, when X carries this topology, that it is the (topological) direct sum of the locally convex spaces (X_α).

Remark 20. Indeed, the direct sum X of locally convex spaces X_α is the inductive limit of the spaces X_A with respect to the mappings ϕ_A, where X_A is the direct sum of the $(X_\alpha)_{\alpha \in A}$ and $\phi_A : X_A \to X$ is defined by $\sum_{\alpha \in A} x_\alpha \to \sum_{\alpha \in A} \phi_\alpha(x_\alpha)$ and A is a finite set of indices α.

To define a nuclear space we need the notion of a nuclear operator.

Definition 21. Let X be a locally convex space and let B be a Banach space. An operator $U : X \to B$ is called a nuclear operator if it can be represented as

$$U(x) = \sum_{n=1}^{\infty} \lambda_n < x_n', x > b_n, \quad for \;\; x \in X,$$

where (x_n') is an equicontinuous sequence of linear functionals on X, (b_n) is a bounded sequence of elements in B and (λ_n) is a sequence of scalars with $\lambda_n \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n$ convergent.

We also need the notion of Minkowski functionals.

Definition 22. A set W in a locally convex space X is

(i)balanced if $x \in W$ and $|\alpha| \leq 1$ implies $\alpha x \in W$, where α is a complex number;

(ii)absorbing if for every $x \in X$, there exists a positive number α such that $\alpha^{-1} x \in W$.

For a balanced, absorbing and convex set W in X the Minkowski functional p_W on X associated to W is defined as

$$p_W(x) = \inf\{\alpha > 0 : \ \alpha^{-1} x \in W\}, \ for \ x \in X.$$

Now, let X be a locally convex space and let W a balanced convex neighborhood of the origin in X. If p_W is the Minkowski functional associated to W, then p_W is a semi-norm on X. Therefore $X_W = X/Ker \ p_W$ is a normed space with the norm $\| \cdot \|_W$ defined by

$$\| \overline{x} \|_W = p_W(x), \ for \ \overline{x} \in X_W,$$

where $x \mapsto \overline{x}$ is the canonical map of X onto X_W. With this norm the completion $\overline{X_W}$ of X_W is a Banach space.

Definition 23. A locally convex space X is called a nuclear space if for any balanced convex neighborhood W of the origin, the canonical mapping $l_W : \ X \rightarrow \overline{X_W}$ is a nuclear operator.

Two important facts about nuclear spaces are as follows.

Fact 1. If a Banach space is also a nuclear space, then it must be finite dimensional.

Fact 2. A nuclear space contains no copy of c_0, the Banach space of all sequence (c_n) of complex number with $\lim c_n = 0$. The norm of c_0 is defined by $|(c_n)| = \sup |c_n|$ for $(c_n) \in c_0$.

There are some good properties for the locally convex spaces that contain no copy of c_0, that is, they have no subspace isomorphic to c_0. For example, they have the B-P (Bessaga-Pelczynski) property.

Definition 24. A locally convex space X has the B-P property if for a sequence (x_n) from X satisfying $\sum | < x', x_n > | < \infty$ for all $x' \in X'$, we have that $\sum x_n$ converges in X.

The following theorem was first given by C.Bessaga and A.Pelczynski(1958) for Banach spaces, and then extended to the case of locally convex spaces (see Li and Bu [21] for references).

Theorem 25.(Bessaga-Pelczynski) A sequentially complete locally convex space X contains no copy of c_0 if and only if X has the B-P property.

In a locally convex space with the B-P property, a series is unconditionally convergent if and only if it is weakly unconditional Cauchy. What about in a general locally convex space? The following theorem answers this question.

Theorem 26. (Orlicz-Pettis) Let X be a locally convex space. A series $\sum x_n$ is unconditionally convergent if and only if it is weakly unconditionally convergent in any topology consistent with the duality between X and X'.

Consequently we shall see that a weakly countably additive X-valued set function on a σ-field S is actually a vector measure.

Remark 27. We make a comment here in order to clarify the concept "topology consistent (or, compatible) with the duality between X and X'." So far we have only dealt with one topology on X, the one determined by semi-norms P, which is called initial topology and it gives the dual space X'. However, there are other topologies

on X which generate the same dual X'. These topologies are called consistent or compatible with the duality between X and X'. Macky's theorem (see Theorem 36.1 in Treves [31]) characterizes these topologies, but for us it suffices to know that both the initial topology and the weak topology are among them.

Remark 28. Combining the Orlicz-Pettis theorem and the Bessaga-Pelczynski theorem we know that the only difference between a general locally convex space and a locally convex space that contains no copy of c_0 can be characterized by an equivalent condition for unconditional convergence of series. For the former type of spaces the equivalent condition would be "weakly unconditional convergence" while for the latter it would be "weakly unconditional Cauchy". And these two conditions coincide if the locally convex space is weakly sequentially complete. Therefore, we have

Theorem 29. A weakly sequentially complete locally convex space contains no copy of c_0, and therefore has the B-P property.

CHAPTER 3
THE BILINEAR INTEGRATION

3.1 Vector Measures

Notions such as total variation, semi-variation and p-semi-variation of a vector measure will be defined in this section. Since our goal is to establish a theory on bilinear integration, we shall focus on vector measures with operator values, $m : S \to L(B, X)$, where B is a Banach space and X is a Hausdorff locally convex space with semi-norms P. We shall extend a result of Drewnowski to obtain an element $x' \in U_p^0$ and an element $b \in B_1$ such that $\lim_{\lambda_p(E)\to 0} \| m \|_p (E) = 0$, where

$$\lambda_p(E) = | < x', m(.)b > |(E), \; E \in S$$

for each p. Therefore we associate each p-semi-variation $\| m \|_p$ with a finite positive measure λ_p. This will later enable us to study the space $L_B^1(m)$ through the space $L_B^1(\Lambda_m)$ with $\Lambda_m = \{\lambda_p : p \in P\}$.

Definition 1. Let T be a set. A field of subsets of T is a non-empty family of subsets of T which contains the void set, the complement of each member and finite unions of its members. If, in addition, a field S of subsets of T has the property that $\bigcup_{n=1}^{\infty} E_n \in S$ whenever $E_n \in S$ for $n = 1, 2, ...$, then S is called a σ-field, and the pair (T, S) is called a measurable space.

Definition 2. Let m be a vector valued or complex valued set function defined on a σ-field S of subsets of a set T. Then m is said to be countably additive if

$$m(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m(E_n),$$

where (E_n) is a sequence of disjoint sets in S. A countably additive vector (complex) valued set function is called a vector (scalar,respectively) measure. Sometimes a scalar measure is simply called a measure.

Definition 3. For a scalar or Banach space valued measure m on the σ-field S of subsets of a set T, the total variation $|m|$ is a set function on S defined by

$$|m|(E) = \sup \sum_{i=1}^{n} |m(E_i)|,$$

where the supremum is taken over all finite collections (E_i) of disjoint sets in S with $E_i \subset E$ and $E \in S$. Total variation sometimes is termed variation. If $|m|(T) < \infty$, we say that m is of bounded variation.

Theorem 4. For a scalar measure m, we have $|m|(E) \leq 4 \sup_{A \subset E} |m(A)|$, where E, A are elements of S. Consequently, a scalar measure is of bounded variation.

For a vector measure m, Theorem 4 does not necessarily hold. If m is Banach space valued, we can construct a finite positive set function which can replace the total variation.

Definition 5. Suppose X is Banach space and $m : S \to X$ is a vector measure. The semi-variation $\| m \|$ of m is defined by

$$\| m \| (E) = \sup | \sum_{i=1}^{n} \alpha_i m(E_i)|,$$

for $E \in S$, where the supremum is taken over all finite collections of complex numbers (α_i) with $|\alpha_i| \leq 1$ and all partitions of E into a finite number of disjoint sets E_i in S.

Theorem 6. The semi-variation $\| m \|$ for a Banach space valued measure m has the following properties:

(i)$\| m \| (F) \leq \| m \| (E)$ if $F \subset E$, and $F, E \in S$;

(ii)$\sup_{F \subset E} |m(F)| \leq \| m \| (E) \leq 4 \sup_{F \subset E} |m(F)| < \infty$, where F and E belong to S;

(iii)$\| m \| (\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \| m \| (E_n)$ for $E_n \in S$ with $n = 1, 2, \ldots$;

(iv)$\| m \| (E) = \sup_{x' \in U^0} | < x', m > |(E)$, where U^0 is the polar for the closed unit ball U in X, and $< x', m >$ is a scalar measure defined by

$$< x', m > (E) = < x', m(E) > \quad for \ E \in S.$$

Part (iv) gives another way to define the semi-variation. By this way it can be easily extended to the case when X is a locally convex space.

Definition 7. Let X be a locally convex space X with semi-norms P. Then for a semi-norm $p \in P$ the p-semi-variation of a vector measure $m : S \to X$ is defined by $\| m \|_p (E) = \sup_{x' \in U_p^0} | < x', m > |(E)$, where $E \in S$ and U_p^0 is the polar for the closed p-unit ball U_p in X.

Consequently, every locally convex space valued measure is associated with a family $\{\| m \|_p \colon p \in P\}$ of finite positive set functions, where $\| m \|_p$ has the properties in Theorem 6 with $\| m \|$ replaced by $\| m \|_p$ and $|m(F)|$ replaced by $p(m(F))$.

Now let us consider vector measures which take values in $L(B, X)$, the space of continuous linear maps of a Banach space B into a locally convex space X with semi-norms P. First of all, we introduce a locally convex topology on $L(B, X)$. For each $p \in P$, define

$$|\varphi|_p = \sup_{b \in B_1} p(\varphi(b))$$

for $\varphi \in L(B, X)$, where $B_1 = \{b \in B : |b| \leq 1\}$. Then $\{|.|_p : p \in P\}$ is a family of semi-norms on $L(B, X)$ that determines a locally convex topology. To simplify the notations, we call the $|.|_p$-semi-variation of m by p-semi-variation and denote it by $\| m \|_p$.

Theorem 8. For a vector measure $m : S \to L(B,X)$ and a semi-norm $p \in P$ on X we have

$$\| m \|_p (E) = \sup\{| < x', m(.)b > |(E) : x' \in U_p^0, b \in B_1\}$$

for $E \in S$, where $U_p^0 \subset X'$ is the polar set for $U_p = \{x \in X : p(x) \leq 1\}$, X' is the dual of X, and $| < x', m(.)b > |$ is the total variation for the scalar measure $< x', m(.)b >$ defined by $< x', m(.)b > (E) =< x', m(E)b >$ for $E \in S$.

Proof.

As in the Banach space case, there are two equivalent ways to define the p-semi-variation $\| m \|_p$ for a locally convex space valued measure, one is given in Definition 7, and the other is given by

$$\| m \|_p (E) = \sup |\sum \alpha_i m(E_i)|_p,$$

where the supremum is taken over all the finite collections (α_i) of complex numbers with $|\alpha_i| \leq 1$ and all the finite partitions (E_i) of E in S. From here we derive the equality by the following calculation:

$$\begin{aligned} \| m \|_p (E) &= \sup_i \sup_{|b| \leq 1} p(\sum \alpha_i m(E_i) b) \\ &= \sup_i \sup_{|b| \leq 1} \sup_{x' \in U_p^0} | < x', \sum \alpha_i m(E_i) b > | \\ &= \sup_{|b| \leq 1} \sup_{x' \in U_p^0} | < x', m(.)b > |(E). \end{aligned}$$

Q.E.D.

Theorem 9. The p-semi-variation $\| m \|_p$ of a vector measure $m : S \to L(B,X)$ has the following properties:

(i) $0 \leq \| m \|_p (F) \leq \| m \|_p (E)$ for $F \in S_E$, where $S_E = \{F \in S : F \subset E\}$ and $E \in S$.

(ii)$\| m \|_p (\cup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \| m \|_p (E_n)$ for $E_n \in S$ $n = 1, 2, \ldots$.

(iii) $\sup_{F\in S_E}|m(F)|_p \leq \| m \|_p (E) \leq 4\sup_{F\in S_E}|m(F)|_p < \infty$ for $E \in S$.

(iv)$\| m \|_p (\liminf E_n) \leq \liminf \| m \|_p (E_n)$

$$\leq \limsup \| m \|_p (E_n) \leq \| m \|_p (\limsup E_n)$$

for $E_n \in S,\ n = 1, 2, \ldots$.

(v)$\lim \| m \|_p (E_n) = \| m \|_p (E)$ if $E_n, E \in S$ and $\lim E_n = E$.

Proof.

(i) and (ii) follow from the definition.

The first inequality in (iii) is true since

$$\| m \|_p (E) \geq \sup\{| < x', m(.)b > |(F) : x' \in U_p^0, b \in B_1, F \in S_E\}$$

$$\geq \sup\{| < x', m(F)b > | : x' \in U_p^0, b \in B_1, F \in S_E\}$$

$$= \sup\{p(m(F)b) : b_1 \in B_1, F \in S_E\}$$

$$= \sup\{|m(F)|_p : F \in S_E\}.$$

And the second one follows from

$$\| m \|_p (E) \leq 4\sup\{| < x', m(F)b > | : x' \in U_p^0, b \in B, F \in S_E\}$$

$$= 4\sup\{|m(F)|_p : F \in S_E\}.$$

For (iv) it suffices to show the first and third inequalities. We know that they are true for finite positive measures. For each $x' \in U_p^0$ and $b \in B_1$,

$$| < x', m(.)b > |(\liminf E_n)| \leq \liminf | < x', m(.)b > |(E_n)$$

$$\leq \liminf \| m \|_p (E_n).$$

Then the first inequality follows. The third inequality follows from (v) and the fact that $\limsup \| m \|_p (E_n) \leq \lim \| m \|_p (\cup_{k\geq n} E_k)$. And (v) is proved by the same argument as in Lewis [19]. Q.E.D.

We know that the semi-variation of a Banach space valued measure is not a measure. However, it is associated with a finite positive measure which is equivalent to the semi-variation in the sense stated in the following theorem.

Theorem 10. (Bartle-Dunford-Schwartz) Let B be a Banach space and $m : S \to B$ be a vector measure. Then there exists a finite positive measure λ on S such that $\lambda(E) \leq \| m \| (E)$ and $\lim_{\lambda(E)\to 0} \| m \| (E) = 0$ for $E \in S$.

In 1970, nearly twenty years after the above theorem was proved, Rybakov improved this result by showing that such a finite positive measure λ can be chosen as a special form. This solved a long posed problem.

Theorem 11. (Rybakov) Let B be a Banach space and $m : S \to B$ be a vector measure. Then there exists an element x' in the dual of B such that $\lambda(E) \leq \| m \| (E)$ and $\lim_{\lambda(E)\to 0} \| m \| (E) = 0$ for $E \in S$, where $\lambda(E) = | < x', m > |(E)$ and $< x', m > (E) = < x', m(E) >$ for $E \in S$.

Definition 12. The x' in the theorem is called a Rybakov functional and

$$\lambda = | < x', m > |$$

is called the Rybakov control measure of m.

Remark 13. We observe that the Rybakov theorem holds for semi-normed spaces.

Proof.

Let B be a semi-normed space with semi-norm p and $m : S \to B$ be a vector measure with p-semi-variation $\| m \|_p$. Let $x \mapsto [x]$ be the canonical map of B onto the quotient space $B/(Kerp)$. Then $B/(Kerp)$ is a normed space with norm p defined by

$$p([x]) = p(x).$$

Let B_C be the completion of $(B/(Kerp), p)$. We define

$$m_C : S \to B_C$$

by $m_C(E) = [m(E)]$ for $E \in S$. Then m_C is a vector measure. By the Rybakov theorem, there exists an element x'_C in the dual of B_C such that $\lambda(E) \leq \| m_C \|_p (E)$ and $\lim_{\lambda(E)\to 0} \| m_C \|_p (E) = 0$ for $E \in S$, where $\lambda = | < x'_C, m_C > |$.

Finally, by defining $x' \in B'$ by $x'(x) = x'_C([x])$ and noting that

$$< x', m > = < x'_C, m_C >$$

and $\| m_C \|_p = \| m \|_p$, we have completed the proof. Q.E.D.

We now introduce some theorems which are contributed by Drewnowski [12] to the development of the Rybakov theorem. The first one concerns operator valued measures.

Theorem 14. (Drewnowski [12]) Let B, X be two Banach spaces and $m : S \to L(B, X)$ be a vector measure. Then there exists an element $b \in B$ and an element $x' \in X'$ such that $\lambda(E) \leq \| m \| (E)$ and $\lim_{\lambda(E)\to 0} \| m \| (E) = 0$. for $E \in S$; where

$$\lambda(E) = | < x', m(.)b > |(E)$$

and $< x', m(.)b > (E) = < x', m(E)b >$ for $E \in S$.

Remark 15. Let Y_1 be a Banach space with norm $|.|$ and Y_2 a semi-normed space with semi-norm p. For a vector measure $m : S \to L(Y_1, Y_2)$, there exists an element $y'_2 \in Y'_2$ and an element $y_1 \in Y_1$ such that $\lambda(E) \to 0$ implies $\| m \|_p (E) \to 0$, where $\lambda(E) = | < y'_2, m(.)y_1 > |(E), E \in S$.

The proof is similar to the proof in Remark 13.

Remark 16. By normalization, y'_2 and y_1 in Remark 15 can be chosen such that $y'_2 \in U^0_p$ and $|y_1| \leq 1$.

By the two remarks above, we have the following theorem which we call the Rybakov-Drewnowski theorem.

Theorem 17. (Rybakov-Drewnowski) Let B be a Banach space and (X, P) be a locally convex space. For a vector measure $m : S \to L(B, X)$ and each $p \in P$, there

exists an $x'_p \in U^0_p$ and a $b_p \in B_1$ such that

$$\lim_{\lambda_p(E)\to 0, E\in S} \| m \|_p (E) = 0,$$

where $\lambda_p(.) = | < x'_p, m(.)b_p > |$ is called the p-Rybakov control measure of m.

The second theorem of Drewnowski is concerned with replacing the range space of vector measure m by certain locally convex spaces. In the locally convex space case, we have to extend the concepts of Rybakov functional and the Rybakov control measure.

Definition 18. Let X be a locally convex space with semi-norms P and let $m : S \to X$ be a vector measure. If there exists an element $x' \in X'$ such that $\lambda(E) \to 0$ if and only if $\| m \|_p (E) \to 0$ for all $p \in P$, where $\lambda(E) = | < x', m > |(E)$ and $E \in S$, then x' is called a Rybakov functional for m and $\lambda = | < x', m > |$ the Rybakov control measure of m.

Theorem 19. (Drewnowski [12]) Let X be a locally convex space and $m : S \to X$ a vector measure. There exists a Rybakov functional for m if X satisfies one of the following conditions.

(i)X is a (LB)-space or, more generally, X is the strict inductive limit of an increasing sequence (X_n) of locally convex spaces such that X_n is closed in X_{n+1} and the Rybakov theorem holds for X_n valued measures for every n.

(ii)X is the locally convex direct sum of spaces for which the Rybakov theorem is valid.

We extend this theorem as follows.

Theorem 20. Let X be an (LB) space. Then for a vector measure $m : S \to L(B, X)$ there exists an element $b \in B_1$ and an element $x' \in X'$ such that $\phi' \in L(B, X)'$ defined by

$$(1) \quad \phi'(\phi) = < x', \phi(b) >$$

for $\phi \in L(B, X)$ is a Rybakov functional for m.

Proof.

Since $A = \{m(E)b : E \in S, b \in B_1\}$ is a bounded subset of X, there exists an n such that $A \subset X_n$. Hence m actually maps S into $L(B, X_n)$. By Theorem 14, there exists a $b \in B_1$ and an $x' \in X'_n$ such that ϕ' defined by (1) is a Rybakov functional for m. Since X_n is a closed subspace of X, by the Hahn-Banach theorem x' can be extended to an element $x' \in X'$. Q.E.D.

3.2 Integrable Functions

Parallel to the hypothesis of finite semi-variation in Brooks and Dinculeanu [6], we need a hypothesis to define the p-upper integrals such that they are finite for integrable functions. The p-upper integrals will serve as semi-norms on the Lebesgue space that will be defined in the next chapter. These p-upper integrals must be defined such that they are continuous with respect to the semi-variations, that is,

$$\lim_{\|m\|_p(E)\to 0} p(m)(f 1_E) = 0,$$

where f is an m-integrable function and $p(m)(f)$ is the p-upper integral of f. The basic notations in this section are as follows: (T, S) is a measurable space, B is a Banach space, X is a Hausdorff locally convex space with semi-norms P, and $m : S \to L(B, X)$ is a vector measure.

Definition 1. A function $f : T \to B$ is called S-simple if there exists a $b_i \in B$ and $E_i \in S$, $i = 1, 2, ..., n$ such that $f = \sum_{i=1}^{n} b_i 1_{E_i}$, where 1_{E_i} is the indicator function of $E_i \in S$, $i = 1, 2, ..., n$. A function $f : T \to B$ is called S-measurable if there exists a sequence (f_n) of S-simple functions such that $f_n(t) \to f(t)$ for all $t \in T$.

Definition 2. For a vector measure $m : S \to L(B, X)$, a set $E \in S$ is called m-null if $\| m \|_p (E) = 0$ for all $p \in P$. The notation "m-a.e." refers to "everywhere but a

m-null set." A function $f : T \to B$ is called <u>m-measurable</u> if there is a sequence (f_n) of S-simple functions such that $f_n \to f$ $m - a.e.$

It is clear that a function $f : T \to B$ is m-measurable if and only if there is a S-measurable function $f_1 : T \to B$ such that $f = f_1$ $m - a.e.$

For scalar S-measurable functions, we have the following results.

<u>Lemma 3.</u> (i) For a bounded, S-measurable function $f : T \to C$, there exists a sequence (f_n) of S-simple functions $f_n : T \to C$ such that $f_n(t) \to f(t)$ uniformly for $t \in T$.

(ii) For an S-measurable $f : T \to C$, there exists a sequence (f_n) of S-simple functions with $|f_n(t)| \leq 2|f(t)|$ and $f_n(t) \to f(t)$ for all $t \in T$.

<u>Proof.</u>

(i) Suppose $|f(t)| \leq M$ for some constant $M > 0$ and all $t \in T$. For each integer $n > 0$, let $\{E_n^i\}_{i=1}^{k_n}$ be a partition of T such that $|f(t) - f(s)| < \frac{1}{2^n}$ for $t, s \in E_n^i$, $1 \leq i \leq k_n$. Pick a point b_n^i from $\{f(t) : t \in E_n^i\}$ and define $f_n = \sum_{i=1}^{k_n} b_n^i 1_{E_n^i}$. Then $|f_n(t) - f(t)| < \frac{1}{2^n}$ for all $t \in T$. It follows that $f_n(t) \to f(t)$ uniformly for $t \in T$.

(ii) For each integer $n > 0$, let $E_n = (|f| \leq n)$. Then $g_n = f 1_{E_n}$ is a bounded S-measurable function. By (i), for $\varepsilon_k \to 0$, there exists a sequence (g_{nk}) of S-simple functions such that $|g_n(t) - g_{nk}(t)| < \varepsilon_k$ for all $t \in T$ and $k > 0$. Now let us define f_{nk} as follows. Let $f_{nk}(t) = g_{nk}(t)$ if $|g_{nk}(t)| > 2\varepsilon_k$ and $f_{nk}(t) = 0$ otherwise. Then, we have that $|g_n(t) - f_{nk}(t)| < \varepsilon_k$ if $|g_{nk}(t)| > 2\varepsilon_k$ and

$$|g_n(t) - f_{nk}(t)| = |g_n(t)| \leq |g_n(t) - g_{nk}(t)| + |g_{nk}(t)| < 3\varepsilon_k$$

if $|g_{nk}(t)| \leq 2\varepsilon_k$. It follows that $f_{nk}(t) \to f_n(t)$, as $k \to \infty$, uniformly for $t \in T$.

Moreover, if $|g_{nk}(t)| > 2\varepsilon_k$, then

$$|g_n(t)| > |f_{nk}(t)| - |g_n(t) - f_{nk}(t)|$$

$$> |g_{nk}| - \varepsilon_k > |g_{nk}(t)| - \frac{1}{2}|g_{nk}(t)|$$
$$= \frac{1}{2}|g_{nk}(t)| = \frac{1}{2}|f_{nk}(t)|.$$

That is, $|f_{nk}(t)| \leq 2|g_n(t)|$. If $|g_{nk}(t)| \leq 2\varepsilon_k$, $|f_{nk}(t)| = 0 \leq 2|g_n(t)|$. Hence $|f_{nk}(t)| \leq 2|g_n(t)|$ for all $t \in T$ and $k > 0$.

Finally, let $f_n(t) = f_{nn}(t)$ for $t \in T$. Then

$$|f_n(t)| \leq 2|g_n(t)| \leq 2|f(t)|$$

for $t \in T$; and

$$|f(t) - f_n(t)| \leq |f(t) - g_n(t)| + |g_n(t) - f_{nn}(t)|$$
$$\leq |f(t) - g_n(t)| + 3\varepsilon_n \to 0$$

for all $t \in T$. Q.E.D.

Remark. It is proved by Dinculeanu that (ii) also holds for B-valued functions (private communication).

In order to define m-integrable functions, we need a hypothesis on m:

Hypothesis. For each $p \in P$, $\sup_{x' \in U_p^0} |m_{x'}|(T) < \infty$, where

$$m_{x'}(.) : S \to B'$$

is a vector measure defined by $m_{x'}(E)b = < x', m(E)b >$ for $E \in S$ and $b \in B$. Let us examine this hypothesis in two special cases:

Case 1. When X is a Banach space, this is the same hypothesis as in Brooks and Dinculeanu [6].

Case 2. When $B = C$, the hypothesis is automatically satisfied.

In the rest of this dissertation, we always assume this hypothesis for the vector measure m.

Definition 4. A m-measurable function $f : T \to B$ is called m-integrable if f is $m_{x'}$-integrable for every $x' \in X'$ and for each $E \in S$, there exists an $x_E \in X$ such that $\int_E f dm_{x'} =< x', x_E >$ for all $x' \in X'$. If f is m-integrable, then $\int_E f dm = x_E$ is called the integral of f on $E \in S$ with respect to m, and $n_f(.) = \int_{(.)} f dm$ is called the indefinite integral of f with respect to m. The integral $\int_T f dm$ is often written as $\int f dm$.

In this definition, the integrability of f with respect to $m_{x'}$ is refered to Dinculeanu [11].

In Case 1 this definition is weaker than the one in Brooks and Dinculeanu [6] while in Case 2 it agrees with the one in Lewis [19] except the condition on f "m-measurable" is replaced by "S-measurable". But we will see that these two conditions make no difference in the resulting Lebesgue space.

Lemma 5. Let $f, g : T \to B$ be m-measurable. and $f = g\ m - a.e.$ If f is m-integrable, then so is g and $\int_E f dm = \int_E g dm$ for all $E \in S$.

Proof.

For every $x' \in X'$, f is $m_{x'}$-integrable. Since $f = g\ m - a.e.$ implies $f = g\ m_{x'} - a.e.$, we see that g is $m_{x'}$-integrable, and for each $E \in S$,

$$\int_E g dm_{x'} = \int_E f dm_{x'} =< x', \int_E f dm > .$$

By definition, g is m-integrable and

$$\int_E g dm = \int_E f dm, \ E \in S.$$

Q.E.D.

Definition 6. For an m-measurable function $f : T \to B$ and a semi-norm $p \in P$ on X we define the p-upper integral as

$$p(m)(f) = \sup_{x' \in U_p^0} \int |f|\ d|m_{x'}|.$$

Before we give some properties of the p-upper integral $p(m)(f)$ of an m-integrable function f, let us introduce a corollary of the Orlicz-Pettis theorem.

Lemma 7. A weakly countably additive set function $n : S \to X$ is countably additive.

Theorem 8. If $f : T \to B$ is m-integrable and $p \in P$, then

(i)$n_f : S \to X$ is a vector measure and $p(m)(f) = \| n_f \|_p (T)$.

(ii)

$$\sup_{E \in S} p(\int_E f dm) \leq p(m)(f)$$
$$\leq 4 \sup_{E \in S} p(\int_E f dm) < \infty.$$

(iii)$\lim_{\|m\|_p(E) \to 0} p(m)(f 1_E) = 0$.

Proof.

The first part of (i) is a consequence of Lemma 7 and the fact that

$$< x', n_f > (.) = \int_{(.)} f dm_{x'}$$

is countably additive. And the second part follows from the definition.

Since $p(m)(f) = \| n_f \|_p (T)$ and $n_f(E) = \int_E f dm$ for $E \in S$, (ii) follows since

$$\sup_{E \in S} p(n_f(E)) \leq \| n_f \|_p (T) \leq 4 \sup_{E \in S} p(n_f(E)).$$

For (iii) we note first that $\| m \|_p (E) = 0$ implies $p(m)(f 1_E) = 0$. Now suppose (iii) is false. Then there exists a positive number ε and a sequence (E_n) of sets in S with $\| m \|_p (E_n) < \frac{1}{2^{n+1}}$ and $p(m)(f 1_{E_n}) > \varepsilon$.

Let $E = \limsup E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$. Then $\| m \|_p (E) = 0$. But applying Theorem 9 in Section 3.1 to vector measure n_f, we have

$$p(m)(f 1_E) = \| n_f \|_p (E)$$

$$= \| n_f \|_p (\limsup E_n)$$

$$\geq \limsup \parallel n_f \parallel_p (E_n)$$

$$= \limsup p(m)(f1_{E_n}) > \varepsilon.$$

This leads to a contradiction. Q.E.D.

As examples of m-integrable functions, we prove that all bounded, m-measurable functions are m-integrable if X is weakly sequentially complete. In fact, this is a corollary of the following theorem.

Theorem 9. If X is a sequentially complete locally convex space such that it does not contain a copy of c_0, then an m-measurable function $f : T \to B$ is m-integrable if and only if it is $m_{x'}$-integrable.

Proof.

It is clear that we only need to show the sufficient part. Without loss of generality, we assume that f is S-measurable.

We first assume that f is a σ-simple function, that is, $f = \sum_{i=1}^{\infty} b_i 1_{A_i}$, where (b_i) is a sequence from B and (A_i) is a sequence of mutual disjoint sets from S. For each set $E \in S$,

$$\sum | < x', m(A_i \cap E)b_i > | \leq \int |f| d|m_{x'}| < \infty.$$

Since X has the B-P property, by Theorem 25 of Chapter 2, it follows that there is an element $x_E \in X$ such that $\sum_{i=1}^{\infty} < x', m(A_i \cap E)b_i > = < x', x_E >$ for every $x' \in X'$. By the definition, f is m-integrable.

For a general S-measurable function f, there exists a sequence (f_n) of σ-simple functions such that $f_n(t) \to f(t)$ uniformly for $t \in T$. For any $\varepsilon > 0$, there exists a positive integer n_0 such that $|f_n(t) - f_k(t)| < \varepsilon$ for $n,\ k \geq n_0$. Thus, for each $E \in S$,

$$p(\int_E f_n - f_k dm) \leq p(m)(f_n - f_k) \leq \varepsilon \sup_{x' \in U_p^0} |m_{x'}|(T).$$

That is, $(\int_E f_n dm)$ is a Cauchy sequence in X. By the sequential completeness of X, we have an element $x_E \in X$ such that $\int_E f_n dm \to x_E$. Therefore

$$\int_E f_n dm_{x'} \to < x', x_E > .$$

On the other hand, $\int_E f_n dm_{x'} \to \int_E f dm_{x'}$ since $f_n(t) \to f(t)$ uniformly for $t \in T$. Hence $\int_E f dm_{x'} = < x', x_E >$ for every $x' \in X'$. This proves the m-integrability of f. Q.E.D.

This theorem supports the assertion that our integrability is defined in a weaker sense than Brooks and Dinculeanu's since bounded functions are not necessarily integrable under their definition even for the case when X is reflexive. So, our definition is a complement to Brooks and Dinculeanu's for bilinear integrals in Banach space case.

Now suppose $B = C$ and X has the B-P property, that is, for $x_n \in X$ with $\sum | < x', x_n > | < \infty$ for every $x' \in X'$, there is an element $x \in X$ such that $\sum x_n = x$. Then, we have the following theorem.

Theorem 10. (Theorem II.5.1 in Kluvanek and Knowles [17]) Suppose X is a locally convex space with the B-P property and $f : T \to C$ is an m-measurable function. Then f is m-integrable if and only if it is $m_{x'}$-integrable for every $x' \in X'$.

In the definition we have two conditions for an m-measurable function $f : T \to B$ to be m-integrable: (i) f is $m_{x'}$-integrable for every $x' \in X'$ and (ii) for each $E \in S$, there exists an element $x_E \in X$ such that $< x', x_E > = \int_E f dm_{x'}$ for all $x' \in X'$. Theorem 9 and Theorem 10 are concerned the cases when (i) implies (ii). By Theorem 25 of Chapter 2, a sequentially complete locally convex space has the B-P property if and only if it contains no copy of space c_0. Hence in the case when X is sequentially complete, Theorem 9 is an extension of Theorem 10.

CHAPTER 4
THE LEBESGUE SPACE

4.1 Definition of the Space

For a vector measure $m : S \to L(B, X)$, let $L_B(m)$ be the linear space of m-integrable functions $f : T \to B$, where S is a σ-field of subsets of a set T, B is a Banach space and X is a Hausdorff locally convex space with semi-norms P. For each $p \in P$, the p-upper integral $p(m)$ of m is a semi-norm on $L_B(m)$. Thus, $L_B(m)$ is a locally convex space, not necessarily Hausdorff, with the topology determined by the family $\{p(m) : p \in P\}$ of semi-norms. We know that $L_B^1(m) = L_B(m)/(\cap_{p \in P} Ker p(m))$ is a Hausdorff locally convex space. More precisely, $L_B^1(m)$ is generated by the following procedure.

First of all we introduce an equivalence relation in $L_B(m)$. Two elements f_1 and f_2 of $L_B(m)$ are called m-equivalent, if $p(m)(f_1 - f_2) = 0$ for every $p \in P$. For $f \in L_B(m)$ we let $[f]_m$ be the class of all elements in $L_B(m)$ which are m-equivalent to f. And we define $L_B^1(m)$ to be the space of all these equivalence classes: $L_B^1(m) = \{[f]_m : f \in L_B(m)\}$. For each $p \in P$, $p(m)$ will be a semi-norm on $L_B^1(m)$ if we set $p(m)([f]_m) = p(m)(f)$. It is clear that $L_B^1(m)$ is a Hausdorff locally convex space with the topology determined by $\{p(m) : p \in P\}$.

<u>Definition 1.</u> The space $L_B^1(m)$ is called the <u>Lebesgue space</u> of the m-integrable functions.

If $B = C$, we write $L^1(m) = L_C^1(m)$.

We make two conventions concerning $L_B^1(m)$:

(i)We will write f instead of $[f]_m$ for elements of $L_B^1(m)$ whenever it is convenient;

(ii)We always assume that $f \in L^1_B(m)$ is S-measurable since for every $[f_1] \in L^1_B(m)$, there exists an S-measurable f such that $f \in [f_1]$ and therefore $[f] = [f_1]$. This explains the remark after the definition of m-integrable functions saying that the conditions "m-measurable" and "S-measurable" make no difference in the resulting Lebesgue space.

We know that each vector measure m is related to a family Λ_m of finite positive measures, the p-Rybakov control measures. One way to study $L^1_B(m)$ is to treat it as a subspace of $L^1_B(\Lambda_m)$ which is defined as follows.

Let Λ be a family of finite scalar measures. An S-measurable function $f : T \to B$ is called Λ-integrable if f is λ-integrable for each $\lambda \in \Lambda$. Denote by $L_B(\Lambda)$ the linear space of all Λ-integrable functions. For each $\lambda \in \Lambda$ we define $p_\lambda(f) = \int |f| d|\lambda|$ for S-measurable functions f. This function on $L_B(\Lambda)$ turns out to be a semi-norm. Thus $L_B(\Lambda)$ becomes a locally convex space with the topology determined by the family $\{p_\lambda : \lambda \in \Lambda\}$ of semi-norms. It is clear that $L_B(\Lambda)$ need not be Hausdorff, but a Hausdorff topology will be obtained by a procedure similar to the one changing $L_B(m)$ into $L^1_B(m)$.

Two elements f_1 and f_2 in $L_B(\Lambda)$ are called Λ-equivalent if $p_\lambda(f_1 - f_2) = 0$ for every $\lambda \in \Lambda$. For $f \in L_B(\Lambda)$, let $[f]_\Lambda$ be the class of all elements in $L_B(\Lambda)$ which are Λ-equivalent to f. Then we define $L^1_B(\Lambda) = \{[f]_\Lambda : f \in L_B(\Lambda)\}$ and $p_\lambda([f]_\Lambda) = p_\lambda(f)$. Thus $L^1_B(\Lambda)$ is a Hausdorff locally convex space with the topology determined by the family $\{p_\lambda : \lambda \in \Lambda\}$ of semi-norms.

When we specify Λ to be the family Λ_m, we will see that $L^1_B(m)$ is continuously embedded in $L^1_B(\Lambda_m)$. For two topological vector spaces Y_1 and Y_2 we say that <u>Y_1 is continuously embedded in Y_2</u> if $Y_1 \subset Y_2$ and the identity mapping from Y_1 to Y_2 is continuous. In this case, we write $Y_1 \hookrightarrow Y_2$.

<u>Lemma 2.</u> For a vector measure $m : S \to L(B, X)$, we have $L^1_B(m) \hookrightarrow L^1_B(\Lambda_m)$.

Proof.

Since

$$| < x', m(.)b > |(E) \leq | < x', m > |(E)$$

for $E \in S$ and $b \in B_1$, we have $p_\lambda(f) \leq p(m)(f)$ for $\lambda = | < x', m(.)b > |$ with $x' \in U_p^0$ and $b \in B_1$, where f is an S-measurable function. The conclusion then follows. Q.E.D.

Remark 3. The fact that $L_B^1(m)$ is continuously embedded in $L_B^1(\Lambda_m)$ is due to the Rybakov-Drewnowski theorem. Let us just examine the case where $B = C$ and X is a Banach space with norm p. If we only use the Bartle-Dunford-Schwartz theorem, we have a control measure λ for m in the sense that $\lambda(E) \leq \| m \|_p (E)$ and $\lim_{\lambda(E)\to 0} \| m \|_p (E) = 0$ for $E \in S$. But we could not conclude that $L^1(m) \subset L^1(\lambda)$. However, if we apply the Rybakov theorem (special case of the Rybakov-Drewnowski theorem) here, we can choose $\lambda = | < x', m > |$ for some $x' \in U_p^0$ and therefore

$$\int |f| d\lambda \leq p(m)(f) < \infty \quad for\ f \in L^1(m).$$

Consequently, $L^1(m) \hookrightarrow L^1(\lambda)$. From this fact we can see why the Rybakov theorem is a useful complement to the Bartle-Dunford-Schwartz theorem.

There are some other cases where the family Λ_m is reduced to be a set which contains a single element. For instance, if X is an (LB) space, then $L_B^1(m) \hookrightarrow L_B^1(\lambda)$, where $\lambda = | < x', m(.)b > |$ with an element $x' \in X'$ and an element $b \in B_1$. In such cases, we can use results available for the space $L_B^1(\lambda)$ to study the space $L_B^1(m)$.

4.2 Convergence Theorems

In integration theory for scalar measures, we have three basic convergence theorems : the Vitali theorem, the Lebesgue theorem and the Beppo-Levi theorem. However, it is pointed out in [6] that the Beppo-Levi theorem is false in general for

vector measure cases. In fact, the Beppo-Levi property turns out to be an important condition to guarantee the weakly sequential completeness of the Lebesgue space. We will discuss this topic in Section 4.5. In this section we will prove the Vitali theorem and the Lebesgue theorem. Note that the Vitali theorem is new for the case where X is a locally convex space, even when $B = C$.

Definition 1. For S-measurable functions f_n and $f : T \to B$, $n = 1, 2, \dots$, we say that $f_n \to f$ in m if for each $\varepsilon > 0$ and $p \in P$, $\| m \|_p (|f_n - f| > \varepsilon) \to 0$, where

$$(|f_n - f| > \varepsilon) = \{t \in T : |f_n(t) - f(t)| > \varepsilon\}.$$

Lemma 2. Suppose $f_n, f : T \to B$ are S-measurable, $n = 1, 2, \dots$.

(i)If $f_n \to f$ $m - a.e.$, then $f_n \to f$ in m.

(ii)If $f_n, f \in L^1_B(m)$, then $f_n \to f$ in $L^1_B(m)$ if and only if $\int_E f_n dm \to \int_E f dm$ in X uniformly for $E \in S$.

Proof.

(i)Since $f_n \to f$ $m - a.e.$, there exists an m-null set $E \in S$ such that $f_n(t) \to f(t)$ for all $t \in T - E$. For each $\varepsilon > 0$, let $E_n = (|f_n - f| > \varepsilon)$ for $n = 1, 2, \dots$. Then we have $\limsup E_n \subset E$. Hence, for each $p \in P$,

$$\limsup \| m \|_p (E_n) \leq \| m \|_p (\limsup E_n)$$

$$\leq \| m \|_p (E) = 0.$$

(ii)This is a consequence of the following inequalities:

$$\sup_{E \in S} p\Big(\int_E f dm\Big) \leq p(m)(f) \leq 4 \sup_{E \in S} p\Big(\int_E f dm\Big).$$

Q.E.D.

Lemma 3. If (x_α) is a Cauchy net in a locally convex space X such that $x_\alpha \to x$ weakly in X for an element $x \in X$, then $x_\alpha \to x$ in X.

Proof.

Suppose X is a locally convex space with semi-norms P. For any $\varepsilon > 0$ and each $p \in P$, there exists an index $\alpha_1 \in A$, the index set for the net (x_α), such that $p(x_\alpha - x_\beta) \leq \varepsilon$ for $\alpha, \beta \in A$ with $\alpha, \beta \geq \alpha_1$. For every $x' \in U_p^0$ and $\alpha, \beta \in A$ with $\alpha, \beta \geq \alpha_1$,

$$\begin{aligned} | < x', x_\alpha - x > | &\leq | < x', x_\alpha - x_\beta > | + | < x', x_\beta - x > | \\ &\leq p(x_\alpha - x_\beta) + | < x', x_\beta - x > | \\ &\leq \varepsilon + | < x', x_\beta - x > |. \end{aligned}$$

Letting $\beta \to \infty$ we get

$$| < x', x_\alpha - x > | \leq \varepsilon$$

for $\alpha \in A$ with $\alpha \geq \alpha_1$. Since $x' \in U_p^0$ is arbitrary, we have

$$p(x_\alpha - x) = \sup_{x' \in U_p^0} | < x', x_\alpha - x > | \leq \varepsilon$$

for $\alpha \in A$ with $\alpha \geq \alpha_1$.

We have shown that $\lim p(x_\alpha - x) = 0$ for every $p \in P$, that is, $x_\alpha \to x$ in X. Q.E.D.

Remark 4. By examining the proof of Lemma 3, we observe the following results on uniform convergence. If $(x_{\alpha,\beta})_{\alpha \in \Delta}$ is a net in a locally convex space X which is Cauchy uniformly for $\beta \in \Omega$, and for each $\beta \in \Omega$, there is an $x_\beta \in X$ such that for each $x' \in X'$, $\lim_\alpha < x', x_{\alpha,\beta} >=< x', x_\beta >$ uniformly for $\beta \in \Omega$, then $\lim_\alpha x_{\alpha,\beta} = x_\beta$ uniformly for $\beta \in \Omega$.

Theorem 5. (Vitali) Suppose X is a sequentially complete locally convex space with semi-norms P, (f_n) is a sequence from $L_B^1(m)$, $f : T \to B$ is an S-measurable function and $f_n \to f$ in m. Then $f \in L_B^1(m)$ and $f_n \to f$ in $L_B^1(m)$ if and only if

$$(1) \quad \lim_{\|m\|_p(E) \to 0, E \in S} p(m)(f_n 1_E) = 0$$

uniformly for $n = 1, 2, \ldots$.

Proof.

First we shall prove the sufficient part. For any $\varepsilon > 0$ and integers $n, k > 0$, define $E_{nk} = (|f_n - f_k| > \varepsilon)$. Then for each $p \in P$,

$$\| m \|_p (E_{nk}) \leq \| m \|_p (|f_n - f| > \frac{\epsilon}{2}) + \| m \|_p (|f_m - f| > \frac{\epsilon}{2}) \to_{n,k\to\infty} 0.$$

By (1), there exists an integer $n_0 > 0$ such that $p(m)(f_i 1_{E_{nk}}) < \varepsilon$ for $n, k > n_0$ and any positive integer i. Therefore we have

$$p(m)(f_n - f_k) \leq p(m)((f_n - f_k)1_{T-E_{nk}}) + p(m)(f_n 1_{E_{nk}}) + p(m)(f_k 1_{E_{nk}})$$

$$\leq \varepsilon \sup_{x' \in U_p^0} |m_{x'}|(T) + 2\varepsilon$$

for $n, k > n_0$, that is, (f_n) is a Cauchy sequence in $L_B^1(m)$. Thus it is also a Cauchy sequence in $L_B^1(m_{x'})$ for each $x' \in X'$. It is clear that $f_n \to f$ in m implies $f_n \to f$ in $m_{x'}$ for each $x' \in X'$. By the Vitali theorem for $L_B^1(m_{x'})$, we have $f \in L_B^1(m_{x'})$ and $f_n \to f$ in $L_B^1(m_{x'})$. It follows that $\int_E f_n dm_{x'} \to \int_E f dm_{x'}$ uniformly for $E \in S$.

On the other hand, for each $E \in S$ and $p \in P$,

$$p(\int_E (f_n - f_k) dm) \leq p(m)(f_n - f_k) \to_{n,k\to\infty} 0,$$

that is, $(\int_E f_n dm)$ is a Cauchy sequence in X, uniformly for $E \in S$. Hence for each $E \in S$, there exists an element $x_E \in X$ such that $\int_E f_n dm \to x_E$. Thus

$$\int_E f_n dm_{x'} = < x', \int_E f_n dm > \to < x', x_E >$$

for all $x' \in X'$.

By definition, $f \in L_B^1(m)$ and $\int_E f dm = x_E$. Moreover, since $(\int_E f_n dm)$ is Cauchy uniformly for $E \in S$ and it converges weakly to $\int_E f dm$ in X uniformly for $E \in S$, by Remark 4, we have the fact that $\int_E f_n dm \to \int_E f dm$ uniformly for $E \in S$. Therefore $f_n \to f$ in $L_E^1(m)$ by Lemma 2.

Now let us prove the necessary part. In order to establish (1), it suffices to observe the following inequality

$$p(m)(f_n 1_E) \leq p(m)(f 1_E) + p(m)(f_n - f)$$

and recall the fact that $\lim_{\|m\|_p(E)\to 0} p(m)(f1_E) = 0$ from Theorem 8 in Section 3.2. Q.E.D.

Corollary 6. (Lebesgue) Suppose X is a sequentially complete locally convex space, $f_n \in L_B^1(m)$, $n = 1, 2, ...$, and $f : T \to B$ is S-measurable and $f_n \to f$ in m. If there is a $g \in L_B^1(m)$ such that $|f_n| \leq |g|$ $m - a.e.$, then $f \in L_B^1(m)$ and $f_n \to f$ in $L_B^1(m)$.

Proof.

By Theorem 5, all we need to show is (1) in Theorem 5. But (1) follows from the following inequality

$$p(m)(f_n 1_E) \leq p(m)(g 1_E)$$

and the fact that $\lim_{\|m\|_p(E)\to 0} p(m)(g1_E) = 0$. Q.E.D.

Corollary 7. Suppose X is a sequentially complete locally convex space and (f_n) is a Cauchy sequence from $L_B^1(m)$. If there exists an S-measurable function $f : T \to B$ such that $f_n \to f$ in m, then $f \in L_B^1(m)$ and $f_n \to f$ in $L_B^1(m)$.

Proof.

By Theorem 5, it suffices to prove (1). But (1) follows from the fact that (f_n) is Cauchy. Q.E.D.

Remark 8. By Lemma 2, Theorem 5 and Corollaries 6 and 7 hold if "$f_n \to f$ in m" is replaced by "$f_n \to f$ $m - a.e.$"

One of applications of the Vitali theorem is to characterize m-integrable functions.

Theorem 9. Suppose X is a sequentially complete locally convex space. An m-measurable function f: $T \to B$ is m-integrable if and only if there exists a sequence (f_n) of S-simple functions such that (f_n) is Cauchy in $L_B^1(m)$, and $f_n \to f$ $m - a.e.$

Proof.

Suppose $f : T \to B$ is m-integrable. There exists a $g \in [f]_m$ such that g is S-measurable. By Lemma 5 in Section 3.2, g is m-integrable. Moreover, by the remark of Lemma 3 in Section 3.2, there exists a sequence (f_n) of S-simple functions such that $|f_n| \leq 2|g|$ and $f_n \to g$ everywhere. By the Lebesgue theorem, $f_n \to g$ in $L^1_B(m)$ and hence (f_n) is Cauchy in $L^1_B(m)$. And it is clear that $f_n \to f$ $m-a.e.$ Conversely, suppose there exists a sequence (f_n) of S-simple functions that is Cauchy in $L^1_B(m)$ and $f_n \to f$ $m-a.e.$ Let E be the set on which the limit of the f_n exists, and $g = f1_E$. Then g is S-measurable and $f_n \to g$ everywhere. By Corollary 7, g is m-integrable and hence so is f. Q.E.D.

4.3 Completeness

Completeness of $L^1(m)$ is discussed in Kluvanek and Knowles [17] where it is characterized in terms of closed vector measures.

Suppose (T, S) is a measurable space, X is a Hausdorff locally convex space with semi-norms P and $m : S \to X$ is a vector measure. Let $S(m) = \{[1_E]_m : E \in S\}$. Then $S(m)$ is a subset of $L^1(m)$.

Definition 1. If $S(m)$ is complete, that is, every Cauchy net in $S(m)$ converges to an element in $S(m)$, then m is called a closed vector measure.

Kluvanek and Knowles prove the following result on the completeness of $L^1(m)$.

Theorem 2. [17] If X is (quasi) complete, then $L^1(m)$ is (quasi) complete if and only if m is closed.

This result has been improved by Ricker [27,28].

Theorem 3. [27] If X is sequentially complete, then $L^1(m)$ is complete if and only if m is closed.

Theorem 4. [28] If the sequential closure $X[m]$ of the linear span of $R(m) = \{m(E) : E \in S\} \subset X$ is sequentially complete, then $L^1(m)$ is complete if and only if m is closed.

The proof of Theorem 2 depends on a key lemma saying that $L^1(\Lambda)$ is complete if and only if its subspace $S(\Lambda) = \{[1_E]_\Lambda : E \in S\}$ is complete and the fact that $S(m) = S(\Lambda_m)$. Combining Theorem 2 and this lemma, we have that $L^1(m)$ is complete if and only if $L^1(\Lambda_m)$ is complete. We then try to extend this result to the bilinear case. However we can only get a partial extension.

Let us return to the bilinear case. Suppose $m : S \to L(B, X)$ is a vector measure and B is a Banach space.

Theorem 5. If X is sequentially complete, then the (sequential, quasi) completeness of $L^1_B(\Lambda_m)$ implies the (sequential, quasi, respectively) completeness of $L^1_B(m)$.

Proof.

First let us assume that $L^1_B(\Lambda_m)$ is sequentially complete. Consider a Cauchy sequence (f_n) from $L^1_B(m)$. Note that it is also a Cauchy sequence in $L^1_B(\Lambda_m)$. Therefore there exists an element $f \in L^1_B(\Lambda_m)$ such that $f_n \to f$ in $L^1_B(\Lambda_m)$. It follows that $f_n \to f$ in $L^1_B(\lambda_p)$ and thus $f_n \to f$ in λ_p for every $\lambda_p \in \Lambda_m$. Consequently $f_n \to f$ in m. Since (f_n) is Cauchy in $L^1_B(m)$, it satisfies the condition of uniform continuity in the Vitali theorem. Hence, by the Vitali theorem, $f \in L^1_B(m)$ and $f_n \to f$ in $L^1_B(m)$. This proves the sequential completeness of $L^1_B(m)$.

Now we assume that $L^1_B(\Lambda_m)$ is complete.

Let (f_α) be a Cauchy net from $L^1_B(m)$. Then it is also a Cauchy net in $L^1_B(\Lambda_m)$. Therefore there exists an element $f \in L^1_B(\Lambda_m)$ such that $f_\alpha \to f$ in $L^1_B(\Lambda_m)$. For each countable subnet (f_{α_k}) of (f_α), by the same argument in the case of sequential completeness, we have $f_{\alpha_k} \to f$ in $L^1_B(m)$. Hence $f_\alpha \to f$ in $L^1_B(m)$. Therefore $L^1_B(m)$ is complete.

Similarly we prove the case of quasi completeness. Q.E.D.

Let us give two special cases where $L^1_B(\Lambda_m)$ is complete, then so is $L^1_B(m)$.

Case 1. X is a Fréchet space.

By a diagonal argument, we have

Lemma 6. If Λ is countable, then $L^1_B(\Lambda)$ is complete.

Consequently,

Theorem 7. If X is a Fréchet space, then $L^1_B(m)$ is complete.

Case 2. X is an (LB) space.

In this case, Λ_m is reduced to a single measure $\lambda = | < x', m(.)b > |$ for some $x' \in X'$ and $b \in B_1$. Thus $L^1_B(\Lambda_m) = L^1_B(\lambda)$ is complete and therefore,

Theorem 8. If X is an (LB) space, $L^1_B(m)$ is complete.

4.4 Integral Representations for Duals

The purpose of this section is to find integral representations for the dual spaces $L^1_B(m)'$ and $L^1_B(\Lambda)'$. First we shall represent each element of $L^1_B(m)'$ by a B'-valued measure. Then by using the Radon-Nikodym theorem (see Dunford and Schwartz [13]), we represent each element of $L^1_B(m)'$ further by a B'-valued function. The set of B'-valued-measure representatives will be denoted by $\Gamma_{B'}(m)$ and the set of B'-valued-function representatives will be denoted by $G_{B'}(m)$. With the topology of uniform convergence on bounded subsets of $L^1_B(m)$, we have an isometric isomorphism between $L^1_B(m)'$ and $\Gamma_{B'}(m)$ and an isometric isomorphism between $L^1_B(m)'$ and $G_{B'}(m)$. Similarly we derive the representations for $L^1_B(\Lambda)'$.

Let $\Gamma_{B'}(m)$ be the set of all vector measures $\mu : S \to B'$ with $|\mu|_I < \infty$ for some finite subset $\{p_i\}_{i \in I} \subset P$, where

$$|\mu|_I = \sup\{\int |f| d|\mu| : f \in L^1_B(m), |f|_I \leq 1\}$$

and $|f|_I = \sup_{i \in I} p_i(m)(f)$.

Let $G_{B'}(m)$ be the set of all S-measurable functions $g : T \to B'$ with $|g|_I < \infty$ for some finite subset $\{p_i\}_{i \in I} \subset P$ where

$$|g|_I = \sup\{\int |fg| d\lambda_I : f \in L^1_B(m), |f|_I \leq 1\},$$

$\lambda_I = \sum_{i \in I} \lambda_{p_i}$ and λ_{p_i} is the p_i-Rybakov control measure of m, $i \in I$. In order to indicate the correspondence between g and λ_I, we will write $(g, \lambda_I) \in G_{B'}(m)$ or $\lambda_I = \lambda_I(g)$.

For every bounded subset $W \subset L^1_B(m)$ we define

(i) $|\varphi|_W = \sup\{|\varphi(f)| : f \in W\}$ for $\varphi \in L^1_B(m)'$;

(ii)$|\mu|_W = \sup\{\int |f| d|\mu| : f \in W\}$ for $\mu \in \Gamma_{B'}(m)$;

(iii)$|g|_W = \sup\{\int |fg| d\lambda_I : f \in W\}$ for $(g, \lambda_I) \in G_{B'}(m)$.

Then (i),(ii),(iii) define semi-norms on $L^1_B(m)'$, $\Gamma_{B'}(m)$ and $G_{B'}(m)$ respectively. When W runs through the family of all bounded subsets of $L^1_B(m)$, we get three families of semi-norms on $L^1_B(m)', \Gamma_{B'}(m)$ and $G_{B'}(m)$ from (i),(ii) and (iii) respectively. Denote by $L^1_B(m)'_s, \Gamma_{B'}(m)_s$ and $G_{B'}(m)_s$ the locally convex spaces $L^1_B(m), \Gamma_{B'}(m)$ and $G_{B'}(m)$ with the topology determined by the corresponding family of semi-norms.

Theorem 1. There is an isometric isomorphism $\varphi \leftrightarrow \mu$ between $L^1_B(m)'_s$ and $\Gamma_{B'}(m)_s$ given by

$$(1) \quad \varphi(f) = \int f d\mu$$

for $f \in L^1_B(m)$.

Proof.

For each element $\varphi \in L^1_B(m)'$, there exists a finite subset $\{p_i\}_{i \in I} \subset P$ and a constant $c > 0$ such that $|\varphi(f)| \leq c|f|_I$. We define a set function $\mu : S \to B'$ by $\mu(E)b = \varphi(1_E b)$ for $b \in B$ and $E \in S$. Then μ is a vector measure of finite variation and (1) holds for S-simple functions $f : T \to B$.

For an element $f \in L^1_B(m)$ there exists a sequence (f_n) of S-simple, B-valued functions such that $f_n(t) \to f(t)$ for every $t \in T$ and $p(m)(f_n - f) \to 0$ for every $p \in P$. It follows that $\varphi(f_n - f) \to 0$.

On the other hand, we have

$$\int |f_n - f_m| d|\mu| \leq 4 \sup_{E \in S} |\int_E f_n - f_m d\mu|$$

$$= 4 \sup_{E \in S} |\varphi((f_n - f_m)1_E)| \leq 4c|f_n - f_m|_I \to_{n,m\to\infty} 0.$$

Thus (f_n) is a Cauchy sequence from $L^1_B(\mu)$ and $f_n(t) \to f(t)$ for $t \in T$. By the Vitali theorem, $f \in L^1_B(\mu)$ and $f_n \to f$ in $L^1_B(\mu)$. Consequently, $\int f_n d\mu \to \int f d\mu$ and (1) holds for f.

Conversely, for an element $\mu \in \Gamma_{B'}(m)$, we define a functional φ on $L^1_B(m)$ by (1). Then $|\varphi(f)| \leq |\mu|_I |f|_I$, for $f \in L^1_B(m)$, implies that $\varphi \in L^1_B(m)'$.

We have established an isomorphism $\varphi \leftrightarrow \mu$ through (1). From (1) we have $|\varphi|_W = |\mu|_W$ for every bounded subset $W \subset L^1_B(m)$. Therefore this isomorphism between $L^1_B(m)'_s$ and $\Gamma_{B'}(m)_s$ is isometric. Q.E.D.

Theorem 2.

(i) Let $(g, \lambda_I) \in G_{B'}(m)$. If a linear functional φ on $L^1_B(m)$ is defined by

$$(1)\ \varphi(f) = \int f g d\lambda_I$$

for $f \in L^1_B(m)$, then $\varphi \in L^1_B(m)'$. Moreover, we have

$$(2)\ |\varphi|_W = |g|_W$$

for every bounded subset $W \subset L^1_B(m)$.

(ii) Conversely, if B' has the Radon-Nikodym property, then for every $\varphi \in L^1_B(m)'$ there exists a $(g, \lambda_I) \in G_{B'}(m)$ satisfying (1) and (2).

Proof.

Let $(g, \lambda_I) \in G_{B'}(m)$ and $\int |fg| d\lambda_I \leq c|f|_I$ for all $f \in L^1_B(m)$. Then the linear functional φ on $L^1_B(m)$ defined by (1) satisfies $|\varphi(f) \leq c|f|_I$. It follows that φ is continuous and therefore $\varphi \in L^1_B(m)'$. It is clear that (1) implies (2).

Conversely, we assume that B' has the Radon-Nikodym property and φ is an element from $L^1_B(m)'$. By Theorem 1, there exists an element μ from $\Gamma_{B'}(m)$ such that

$$(3) \quad \varphi(f) = \int f d\mu.$$

Suppose there exists a positive constant c and a finite subset $\{p_i\}_{i \in I} \subset P$ such that $|\varphi(f)| \leq c|f|_I$. By (3) we have the fact that $\mu \ll \lambda_I$, where $\lambda_I = \sum_{i \in I} \lambda_i$, and λ_i is the p_i-Rybakov control measure of m. Since B' has the Radon-Nikodym property, there exists a λ_I-integrable, B' valued function g such that $\mu(E) = \int_E g d\lambda_I$. Thus we have (1), (2) and $(g, \lambda_I) \in G_{B'}(m)$. Q.E.D.

This theorem tells that if B' has the Radon-Nikodym property, there is an isometric isomorphism $\varphi \leftrightarrow g$ between $L^1_B(m)'_s$ and $G_{B'}(m)_s$ given by (1).

Remark 3. (i) When X is an (LB) space, we know that Λ_m is reduced to a single λ. In this case, we can use λ to characterize $L^1_B(m)'$. In fact, by the same argument for Theorem 2, we show that if X is an (LB) space, λ_I in Theorem 2 can be replaced by λ and $G_{B'}(m)$ can be replaced by $G_{B'}(m, \lambda)$, the set of all S-measurable functions $g : T \to B'$ such that there exists a finite subset $\{p_i\}_{i \in I} \subset P$ satisfying $|g|_{I,\lambda} \leq \infty$, where

$$|g|_{I,\lambda} = \sup\{\int |fg| d\lambda : f \in L^1_B(m), |f|_I \leq 1\}.$$

(ii) When X is a Banach space and B' has the Radon-Nikodym property, it is proved in Brooks and Dinculeanu [6] that $L^1_B(m)$ is isometric isomorphic to a space

$$G(m, B') = \{g \in L^1_{B'}(\lambda) : h|g| \in L^1(\lambda) \; for \; every \; h \in L^1(m)\},$$

where λ is the control measure of m. According to Theorem 2, $L^1_B(m)$ is isometric isomorphic to

$$G_{B'}(m) = \{g \in L^1_{B'}(\lambda) : \int |< g, f >| d\lambda \leq c \parallel m \parallel (f) \ for\ every\ f \in L^1_B(m)\}$$

where $c > 0$ is some constant and $\parallel m \parallel (f)$ is the upper integral of f. In fact, although our Lebesgue space is different from Brooks and Dinculeanu's, but we have $G_{B'}(m) = G(m, B')$, that is, these two Lebesgue spaces have the same dual!

Proof.

If $g \in G_{B'}(m)$ and $h \in L^1(\lambda)$, for $b \in B_1$, $bh \in L^1_B(\lambda)$. And

$$\int |h||g| d\lambda \leq sup_{b \in B_1} \int |< g, bh >| d\lambda$$

$$\leq c \parallel m \parallel (bh) \leq c \parallel m \parallel (h) < \infty.$$

Hence $g \in G(m, B')$ and therefore $G_{B'}(m) \subset G(m, B')$. Conversely, let $g \in G(m, B')$. Since $h|g| \in L^1(\lambda)$ for every $h \in L^1(m)$, by Lemma 5.1 in Brooks and Dinculeanu [6],

$$c = \sup\{\int |hg| : h \in L^1(\lambda), \parallel m \parallel (h) \leq 1\} < \infty.$$

Then for $f \in L^1_B(m)$, $|f| \in L^1(m)$, consequently,

$$\int |< g, f >| d\lambda \leq \int |f||g| d\lambda$$

$$\leq c \parallel m \parallel (f).$$

It follows that $g \in G_{B'}(m)$ and $G(m, B') \subset G_{B'}(m)$. Q.E.D.

Similarly, we will find the representations for $L^1_B(\Lambda)'$, where Λ is a family of finite scalar measures on S.

First of all, we define a set $\Gamma_{B'}(\Lambda)$ of B' valued vector measures as follows. A vector measure $\mu : S \rightarrow B'$ belongs to $\Gamma_{B'}(\Lambda)$ if and only if there exists a constant $c > 0$ and a finite subset $\{\lambda_i\}_{i \in I} \subset \Lambda$ such that $|\mu| \leq c\lambda_I$, where $\lambda_I = \sum_{i \in I} |\lambda_i|$.

Then, we define a set $G_{B'}(\Lambda)$ of B' valued functions. An S-measurable function $g : T \to B'$ is an element of $G_{B'}(\Lambda)$ if it satisfies the following condition: there exists a positive constant $c > 0$ and a finite subset $\{\lambda_i\}_{i\in I} \subset \Lambda$ such that $|g|_I < \infty$, where

$$|g|_I = \sup\{\int | < g, f > |d\lambda_I : f \in L^1_B(\Lambda), |f|_I \leq 1\}$$

$\lambda_I = \sum_{i\in I} \lambda_i$ and $|f|_I = \sup_{i\in I} \int |f| d\lambda_i$. In order to see the connection between g and λ_I we sometimes write $(g, \lambda_I) \in G_{B'}(\Lambda)$ or $\lambda_I = \lambda_I(g)$.

Next we equip the three spaces, $L^1_B(\Lambda)'$, $\Gamma_{B'}(\Lambda)$ and $G_{B'}(\Lambda)$, with the bounded convergence topology, that is, the topology determined by the family $\{|.|_W : W \subset L^1_B(\Lambda)$ bounded $\}$ of semi-norms defined as follows.

(i)For $\varphi \in L^1_B(\Lambda)'$, $|\varphi|_W = \sup\{| < \varphi, f > | : f \in W\}$;

(ii)For $\mu \in \Gamma_{B'}(m)$, $|\mu|_W = \sup\{\int |f| d|\mu| : f \in W\}$;

(iii)For $g \in G_{B'}(\Lambda)$, $|g|_W = \sup\{\int | < g, f > |d\lambda_I : f \in W\}$.

With these topologies, these three spaces become three locally convex spaces which are denoted by $L^1_B(\Lambda)'_s, \Gamma_{B'}(\Lambda)_s$ and $G_{B'}(\Lambda)_s$ respectively.

Finally by the similar arguments to those for the representations of $L^1_B(m)'$, we have the following results.

Theorem 4. There exists an isometric isomorphism $\varphi \leftrightarrow \mu$ between $L^1_B(\Lambda)'_s$ and $\Gamma_{B'}(\Lambda)_s$ given by $< \varphi, f >= \int f d\mu$ for $f \in L^1_B(\Lambda)$. Moreover, if B' has the Radon-Nikodym property, then there also exists an isometric isomorphism $\varphi \leftrightarrow g$ between $L^1_B(\Lambda)'_s$ and $G_{B'}(\Lambda)_s$ given by $< \varphi, f >= \int < g, f > d\lambda_I$ for $f \in L^1_B(\Lambda)$, where $\lambda_I = \lambda_I(g)$.

Remark 5. The difference between the construction of $\Gamma_{B'}(\Lambda)$ and $\Gamma_{B'}(m)$ is due to the difference between λ_p and $\| m \|_p$. For each finite subset $\{\lambda_i\}_{i\in I} \subset \Lambda$, $\lambda_I = \sum_{i\in I} \lambda_i$ is a measure since each λ_i is a measure, therefore $|\mu| \leq c\lambda_I$ is equivalent to $\int |f| d|\mu| \leq c \int |f| d\lambda_I$ for $f \in L^1_B(\Lambda)$. However, for each $p \in P$, $p(m)$ is not a

countably additive set function, hence $|\mu|(E) \leq c \sup_{i\in I} \| m \|_{p_i} (E)$ for $E \in S$ does not imply $\int |f| d|\mu| \leq c \sup_{i\in I} p_i(m)(f)$ for $f \in L^1_B(\Lambda)$. So we have to keep this more complicated inequality in the definition of elements $\mu \in \Gamma_{B'}(m)$.

Remark 6. In the case when $B = C$, a representation of $L^1(\Lambda)'$ is given by Theorem III.2.1 in the book of Kluvanek and Knowles [17]. It says that there is an isomorphism $\varphi \leftrightarrow \mu$ between $L^1(\Lambda)'$ and the space Γ_Λ given by $< \varphi, f >= \int f d\mu$ for $f \in L^1(\Lambda)$, where the space Γ_Λ is defined as follows: a scalar measure μ on S belongs to Γ_Λ if and only if there exists a positive constant k and a measure $\lambda \in \Lambda$ such that $|\mu| \leq k|\lambda|$. The difference between $\Gamma(\Lambda)$ and Γ_Λ is obvious. In $\Gamma(\Lambda)$ each element μ is dominated by a finite number of elements from Λ while in Γ_Λ, it is only dominated by a single element of Λ. In fact, it is a mistake to use Γ_Λ instead of $\Gamma(\Lambda)$. This mistake is due to a misunderstanding of the topology of $L^1(\Lambda)$. Since the topology of $L^1(\Lambda)$ is determined by the family $\{p_\lambda : \lambda \in \Lambda\}$ of semi-norms defined by $p_\lambda(f) = \int |f| d\lambda$, then

$$\bigcup \{f : f \in L^1(\Lambda), \sup_{i\in I} \int |f| d|\lambda_i| < \varepsilon\},$$

or equivalently,

$$\bigcup \{f : f \in L^1(\Lambda), \int |f| d\lambda_I < \varepsilon\}$$

forms a basis of neighborhoods of zero in $L^1(\Lambda)$, where the union is taken over all finite subsets $\{\lambda_i\}_{i\in I} \subset \Lambda$ and all positive numbers ε, and $\lambda_I = \sum_{i\in I} \lambda_i$. Consequently, $\varphi \in L^1(\Lambda)'$ if and only if there exists a positive constant c and a finite subset $\{\lambda_i\}_{i\in I} \subset \Lambda$ such that $| < \varphi, f > | \leq c \int |f| d\lambda_I$. In Kluvanek and Knowles [17], however, the family $\{f : f \in L^1(\Lambda), \int |f| d\lambda < \varepsilon\}, \lambda \in \Lambda$ and $\varepsilon > 0$, is taken to be the basis of neighborhoods of zero in $L^1(\Lambda)$ and therefore $\varphi \in L^1(\Lambda)'$ if and only if $< \varphi, f > \leq k \int |f| d\lambda$ for some $\lambda \in \Lambda$ and $k > 0$.

Remark 7. Note that $g \in G_{B'}(m) \bigcap G_{B'}(\Lambda)$ if $g : T \to B'$ is a bounded S-measurable function. Hence g corresponds an element $\varphi_1 \in L^1_B(m)'$ and an element $\varphi_2 \in L^1_B(\Lambda)'$ through the corresponding isomorphism.

4.5 Weakly Sequential Completeness and Compactness

We have seen that the completeness of $L^1_B(\Lambda_m)$ implies that of $L^1_B(m)$. Now we shall see that this implication is also true for weakly sequential completeness under certain conditions. Consequently, if B is reflexive, X is a Fréchet space or an (LB) space and m has the Beppo-Levi property, then $L^1_B(m)$ is weakly sequentially complete. This result is parallel to the one in Brooks and Dinculeanu [6] when X is a Banach space. The result on weakly sequential compactness in Brooks and Dinculeanu [6] is also converted to the case when X is either a Fréchet space or an (LB) space.

Let $M_B(S)$ be the set of all S-measurable functions $f : T \to B$. For a vector measure $\mu : S \to B'$ of finite total variation we define

$$|\mu|_p = \sup\{\int |f| d|\mu| : f \in M_B(S), p(m)(f) \leq 1\}$$

for $p \in P$.

Lemma 1. For $f \in M_B(S)$ we have

(i) $p(m)(f) = \sup\{\int |f| d|\mu| : |\mu|_p \leq 1\}$ and

(ii) $p(m)(f) < \infty$ if $f \in \bigcap_{|\mu|_p \leq 1} L^1_B(\mu)$.

Proof.

For every $x' \in U^0_p$, $\int |f| d|m_{x'}| \leq p(m)(f)$ implies $|m_{x'}|_p \leq 1$. It follows that

$$p(m)(f) = \sup_{x' \in U^0_p} \int |f| d|m_{x'}| \leq \sup_{|\mu|_p \leq 1} \int |f| d|\mu|.$$

On the other hand, for a vector measure $\mu : S \to B'$ with $|\mu|_p < \infty$, we have $\int |f| d|\mu| \leq |\mu|_p p(m)(f)$. Hence

$$\sup_{|\mu|_p \leq 1} \int |f| d|\mu| \leq p(m)(f).$$

We have shown that (i) holds.

In order to prove (ii), for a contradiction, suppose $p(m)(f) = \infty$ for $f \in \bigcap_{|\mu|_p \leq 1} L^1_B(\mu)$. By (i) there exists a sequence (μ_n) of B'-valued vector measures with $|\mu_n|_p \leq 1$ such that $\int |f| d|\mu_n| > n2^n$ for every integer $n > 0$. For a fixed $b' \in B'$ with $|b'| = 1$ we define

$$\mu(.) = \sum_{n=1}^{\infty} \frac{|\mu_n|(.)b'}{2^n(|\mu_n|(T) + 1)}.$$

Then $\mu : S \to B'$ is a vector measure with $|\mu|_p \leq 1$. But $\int |f| d|\mu| = \infty$ contradicts the fact that $f \in L^1_B(\mu)$. Q.E.D.

Let $F_B(m)$ be the set of all functions in $M_B(S)$ with $p(m)(f) < \infty$ for all $p \in P$. Then $L^1_B(m) \subset F_B(m)$.

We adopt the following notion from Brooks and Dinculeanu [6].

Definition 2. A vector measure $m : S \to L(B, X)$ is said to have the Beppo-Levi property if $L^1_B(m) = F_B(m)$.

Lemma 3. Suppose B' has the Radon-Nikodym property, m has the Beppo-Levi property and $f \in M_B(S)$. Then $f \in L^1_B(m)$ if and only if $fg \in L^1(\lambda_I)$ for every $(g, \lambda_I) \in G_{B'}(m)$.

Proof.

If $f \in L^1_B(m)$ and $(g, \lambda_I) \in G_{B'}(m)$, we have

$$\int |fg| d\lambda_I \leq |g|_I |f|_I \leq \infty.$$

It follows that $fg \in L^1(\lambda_I)$.

Conversely, for each $p \in P$ if $\mu : S \to B'$ is a vector measure with $|\mu|_p \leq 1$, then $\mu \in \Gamma_{B'}(m)$. Consequently there exists a $(g, \lambda_I) \in G_{B'}(m)$ such that $\mu(.) = \int_{(.)} g d\lambda_I$. Hence

$$\int |f| d|\mu| = \int |fg| d\lambda_I < \infty$$

implies $f \in L^1_B(\mu)$. By Lemma 1, it follows that $f \in F_B(m) = L^1_B(m)$. Q.E.D.

With the above lemma, we are able to prove:

Theorem 4. Suppose B' has the Radon-Nikodym property and m has the Beppo-Levi property. Then $L^1_B(m)$ is weakly sequentially complete if $L^1_B(\Lambda_m)$ is weakly sequentially complete.

Proof.

Let (f_n) be a weak Cauchy sequence in $L^1_B(m)$. Then it is also a weak Cauchy sequence in $L^1_B(\Lambda_m)$. Since we assume that $L^1_B(\Lambda_m)$ is weakly sequentially complete, there exists an element $f \in L^1_B(\Lambda_m)$ such that

$$(1) \;\; f_n \to f \textit{ weakly in } L^1_B(\Lambda_m).$$

For each $(g, \lambda_I) \in G_{B'}(m)$, there exists a $c > 0$ such that $\int |f_n g| d\lambda_I \leq c \sup_{i \in I} p_i(m)(f_n)$ for $n = 1, 2, \ldots$. It follows that $(f_n g)$ is bounded in $L^1(\lambda_I)$ since (f_n) is bounded in $L^1_B(m)$. Moreover, for every $E \in S$ the sequence $(\int_E f_n g d\lambda_I)$ is Cauchy by the representation of $L^1_B(m)'$ through $G_{B'}(m)$ and the fact that $g 1_E \in G_{B'}(m)$. Consequently, $(f_n g)$ is a weak Cauchy sequence in $L^1(\lambda_I)$ and therefore there exists an element $f_g \in L^1(\lambda_I)$ such that

$$(2) \;\; f_n g \to f_g \textit{ weakly in } L^1(\lambda_I).$$

For every positive integer n we let $E_n = (|g| \leq n)$. Thus $g 1_{E_n \cap E} \in G_{B'}(m)$ for every $E \in S$ since $g 1_{E_n \cap E}$ is bounded. By (1) we have

$$(3) \;\; \int_{E_n \cap E} f_n g d\lambda_I \to \int_{E_n \cap E} f g d\lambda_I \textit{ for every } E \in S.$$

Meanwhile, since $1_{E_n \cap E} \in L^\infty(\lambda_I)$, by (2), we have

$$(4) \quad \int_{E_n \cap E} f_n g d\lambda_I \to \int_{E_n \cap E} f_g d\lambda_I \text{ for every } E \in S.$$

Hence we have $fg = f_g \ \lambda_I - a.e.$ by comparing (3) and (4) and noting that $T = \cup_{n=1}^{\infty} E_n$. Therefore $fg \in L^1(\lambda_I)$ and $\int f_n g d\lambda_I \to \int fg d\lambda_I$. By Lemma 3 it follows that $f \in L_B^1(m)$ and $f_n \to f$ weakly in $L_B^1(m)$. Q.E.D.

Now the question is : when is $L_B^1(\Lambda_m)$ weakly sequentially complete? A quick answer is when Λ_m is countable, or equivalently, X is metrizable and both B and B' have the Radon-Nikodym property. First let us prove a lemma.

Lemma 5. Let B be a weakly sequentially complete Banach space such that B and B' have the Radon-Nikodym property. If λ is a real finite measure, then $L_B^1(\lambda)$ is weakly sequentially complete.

Proof.

Let (f_n) be a weak Cauchy sequence from $L_B^1(\lambda)$. Since B' has the Radon-Nikodym property, $L_B^1(\lambda)' = L_{B'}^\infty(\lambda)$ (see Diestel and Uhl [10]). For every $x' \in X'$ and $E \in S$,

$$< x', \int_E f_n d\lambda >= \int < (x' 1_E), f_n > d\lambda$$

converges. Thus $(\int_E f_n d\lambda)$ is weak Cauchy in B and therefore weakly convergent to an element in B, denoted by $\mu(E)$. This defines a set function $\mu : S \to B$ which is weakly countably additive. By the Orlicz-Pettis theorem, μ is a vector measure. Then by the Vitali-Hahn-Saks theorem (see Dunford and Schwartz [13]), we have $\mu \ll \lambda$, that is, μ is λ-continuous. Since B has the Radon-Nikodym property, there exists an element $f \in L_B^1(\lambda)$ such that $\mu(.) = \int_{(.)} f d\lambda$ Hence, $\int_E f_n d\lambda \to \int_E f d\lambda$ for every $E \in S$. It follows that $\int f_n g d\lambda \to \int fg d\lambda$ for all $g \in L_{B'}^\infty(\lambda)$, that is, $f_n \to f$ weakly in $L_B^1(\lambda)$. Q.E.D.

Since a reflexive Banach space is weakly sequentially complete and both the space itself and its dual have the Radon-Nikodym property (see Diestel and Uhl [10]), we have the following corollary.

Corollary 6. If B is a reflexive Banach space, then $L^1_B(\lambda)$ is weakly sequentially complete.

Theorem 7. If Λ is a countable family of finite positive measures on S and B is a reflexive Banach space, then $L^1_B(\Lambda)$ is weakly sequentially complete.

Proof.

Without loss of generality we assume that all the elements of Λ are ordered set-wisely as $0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_k \leq ....$ Thus for each integer $k > 0$ there exists an S-measurable function $h_k : T \to R$ with $0 \leq h_k \leq 1$ such that

$$(1) \;\; \lambda_k = \int h_k d\lambda_{k+1}.$$

Let (f_n) be a weak Cauchy sequence in $L^1_B(\Lambda)$. Then it is a weak Cauchy sequence in $L^1_B(\lambda_k)$ for each integer $k > 0$ since

$$L^1_B(\Lambda) \subset L^1_B(\lambda_k) \; implies \; L^1_B(\lambda_k)' \subset L^1_B(\Lambda)'.$$

By the weakly sequential completeness of $L^1_B(\lambda_k)$, there exists an element $g_k \in L^1_B(\lambda_k)$ such that

$$(2) \;\; \int f_n h d\lambda_k \to \int g_k h d\lambda_k$$

for all $h \in L^\infty_{B'}(\lambda_k)$. Replacing k with $k+1$ in (2) we have

$$(3) \;\; \int f_n h d\lambda_{k+1} \to \int g_{k+1} h d\lambda_{k+1}$$

for all $h \in L^\infty_{B'}(\lambda_{k+1})$. Since

$$L^1_B(\lambda_{k+1}) \subset L^1_B(\lambda_k) \; and \; L^\infty_{B'}(\lambda_{k+1}) \supset L^\infty_{B'}(\lambda_k),$$

for $h \in L^{\infty}_{B'}(\lambda_k)$, we have $hh_k \in L^{\infty}_{B'}(\lambda_{k+1})$. By (3) we have

$$\int f_n hh_k d\lambda_{k+1} \to \int g_{k+1} hh_k d\lambda_{k+1},$$

that is,

$$(4) \quad \int f_n h d\lambda_k \to \int g_{k+1} h d\lambda_k$$

for all $h \in L^{\infty}_{B'}(\lambda_k)$. By (2) and (4) we have $g_{k+1} = g_k \ \lambda_k - a.e.$. Let $f(t) = \lim_{k\to\infty} g_k(t)$ if the limit exists for $t \in T$ and $f(t) = 0$ otherwise. Note that $f = g_k \ \lambda_k - a.e.$ for every $k > 0$. Therefore $f \in L^1_B(\Lambda)$ and $f_n \to f$ weakly in $L^1_B(\lambda_k)$ for every $k > 0$.

For each $\varphi \in L^1_B(\Lambda)'$, there exists a finite subset $\{\lambda_i\}_{i\in I} \subset \Lambda$ and a constant $c > 0$ such that $|\varphi(f)| \leq c \sup_{i\in I} \int |f| d\lambda_i$. Let $\lambda_k = \max_{i\in I} \lambda_i$. Then $\lambda_k \in \Lambda$ and we have $|\varphi(f)| \leq c \int |f| d\lambda_k$. Therefore $\varphi \in L^1_B(\lambda_k)'$. Consequently, $\varphi(f_n - f) \to 0$. That is, $L^1_B(\Lambda)$ is weakly sequentially complete. Q.E.D.

The author would like to thank Prof. Murali Rao for pointing out the idea of this proof.

Corollary 8. If B is a reflexive Banach space, X is a Fréchet space or an (LB) space and m has the Beppo-Levi property, then $L^1_B(m)$ is weakly sequentially complete.

Proof.

By Theorem 4, we only need to check the weakly sequential completeness of $L^1_B(\Lambda_m)$. If X is a Fréchet space, Λ_m is countable and $L^1_B(\Lambda_m)$ is weakly sequentially complete by Theorem 7. If X is an (LB) space, Λ_m is reduced to λ and $L^1_B(\Lambda_m) = L^1_B(\lambda)$ is weakly sequentially complete by Corollary 6. Q.E.D.

Let us discuss this result in two cases. First, when X is a Banach space, it is parallel to the corresponding theorem in Brooks and Dinculeanu [6]. Second, when $B = C$, Corollary 8 implies the following result.

Corollary 9. If X is a Fréchet space or an (LB) space which contains no copy of c_0 (for example, if X is a nuclear Fréchet space), then $L^1(m)$ is weakly sequentially complete.

Proof.

By Theorem 7, $L^1(\Lambda_m)$ is weakly sequentially complete. Thus, by Theorem 4, it suffices to show that m has the Beppo-Levi property. First we note that

$$L^1(m) \subset F(m) \subset \bigcap_{x' \in X'} L^1(m_{x'}).$$

Since X does not contain a copy of c_0, by the B-P theorem (Theorem 25 in Chapter 2), it has the B-P property. Hence by Theorem 10 in Section 3.2,

$$L^1(m) = \bigcap_{x' \in X'} L^1(m_{x'}),$$

therefore $L^1(m) = F(m)$ and the claim is proved. Q.E.D.

Now let us turn to weakly sequential compactness. First we have sufficient conditions for weak compactness on $L^1_B(\Lambda)$ when Λ is countable (equivalent to weakly sequential compactness in this case). Let us recall a result on the case when Λ contains only one element λ.

Lemma 10. (Brooks-Dinculeanu [7]) If B' has (respectively B' and B have) the Radon-Nikodym property, then K is conditionally (relatively) weakly compact if and only if K satisfies

(1) K is bounded;

(2) $\lim_{\lambda(E)\to 0} \int_E f d\lambda = 0$ uniformly for $f \in K$;

(3) For each $E \in S$, $\{\int_E f d\lambda : f \in K\}$ is a conditionally (relatively) weakly compact set in B.

Next we assume that Λ is countable.

Theorem 11. Suppose Λ is countable and B' (and B) has (have) the Radon-Nikodym property, then K is conditionally (relatively) weakly compact if K satisfies

(1)' K is bounded;

(2)' For each $\lambda \in \Lambda$, K satisfies (2);

(3)' For each $\lambda \in \Lambda$ and $E \in S$, $\{\int_E f d\lambda : f \in K\}$ is a conditionally (relatively) weakly sequentially compact set in B.

The proof is completed by Lemma 10 and a diagonal process.

We then have the sufficient conditions for the weakly sequential compactness for $L^1_B(m)$.

Theorem 12. Suppose B is a reflexive Banach space and X is a Fréchet space or an (LB) space. If a subset K in $L^1_B(m)$ satisfies:

(i)K is bounded;

(ii)for each $p \in P$, $\lim_{\|m\|_p(E)\to 0} p(m)(f 1_E) = 0$ uniformly for $f \in K$,

then K is conditionally weakly sequentially compact. If in addition, m has the Beppo-Levi property, then K is relatively weakly sequentially compact.

Proof.

First suppose X is a Fréchet space. Note that (i) and (ii) imply (1)' and (2)' and (1)' in turn implies (3)' since B is reflexive. By Theorem 11 we know that K is relatively weakly sequentially compact in $L^1_B(\Lambda_m)$. For any sequence (f_n) from K, there exists a subsequence (f_n) (for simplification, assume the subsequence is (f_n)) and an element $f \in L^1_B(\Lambda_m)$ such that $f_n \to f$ weakly in $L^1_B(\Lambda_m)$.

Let (g, λ_I) be an element of $G_{B'}(m)$. Then $g \in L^1_{B'}(\lambda_I)$ and $\lim_{k\to\infty} \lambda_I(E_k) = 0$, where $E_k = (|g| > k)$, since

$$\lambda_I = \sum_{i\in I} \lambda_{p_i}, \; and \; \lim_{k\to\infty} \| m \|_{p_i} (E_k) = 0 \; for \; i \in I.$$

By (ii), for $\varepsilon > 0$, there exists a k such that $p_i(m)(f_n 1_{E_k}) < \varepsilon$ for all $i \in I$ and positive integers n. It follows that $\int_{E_k} |f_n| d\lambda_{p_i} < \varepsilon$ for all $i \in I$ and $n > 0$.

On the other hand, $g1_{E_k^c}$ is bounded, hence $(g1_{E_k^c}, \lambda_I)$ is an element of $G_{B'}(\Lambda_m)$. Thus there exists an $n_0 > 0$ such that if $n, l > n_0$ then

$$|\int (f_n - f_l) g 1_{E_k^c} d\lambda_I| < \varepsilon.$$

Therefore,

$$\begin{aligned} &|\int (f_n - f_l) g d\lambda_I| \\ &\leq |\int (f_n - f_l) g 1_{E_k^c} d\lambda_I| + \int_{E_k} |f_n| d\lambda_I + \int_{E_k} |f_l| d\lambda_I \\ &\leq \varepsilon + 2 n_I \varepsilon, \end{aligned}$$

where n_I is the number of elements in I. We have shown that (f_n) is a weak Cauchy sequence in $L^1_B(m)$. Hence K is conditionally weakly sequentially compact.

If in addition, m has the Beppe-Levi property, $L^1_B(m)$ is weakly sequentially complete and (f_n) is weakly convergent in $L^1_B(m)$. This shows that K is relatively weakly sequentially complete.

Now suppose X is an (LB) space and $\Lambda_m = \{\lambda\}$. Note that (i) and (ii) imply (1) and (2), and (1) in turn implies (3) since B is reflexive. By Lemma 10, K is a conditionally weakly compact set in $L^1_B(\lambda)$. By the same argument as in the Fréchet space case, with λ_I replaced by λ, we prove the weakly sequential compactness of K in $L^1_B(m)$. Q.E.D.

CHAPTER 5
APPLICATIONS TO CONTROL SYSTEMS

5.1 Remarks on Vector Measures and Control Systems by Kluvanek and Knowles

We mentioned in Section 4.4 that there is a mistake in Theorem III.2.1 of Kluvanek and Knowles [17]. Consequently some other theorems of Kluvanek and Knowles [17] contain mistakes also when Theorem III.2.1 is applied in their proofs. The purpose of this section is to correct these mistakes. These corrected theorems form a foundation for the upcoming topics on control systems.

In this section we shall try to use the same notation as in Kluvanek and Knowles [17]. Let $ca(S)$ be the set of all measures $\mu : S \to R$, where S is a σ-field of subsets of a set T and R is the real number field. For $\Lambda \subset ca(S)$, $\tau(\Lambda)$ is the topology on $L^1(\Lambda)$ determined by the family of semi-norms $\{p_\lambda : \lambda \in \Lambda\}$, where

$$p_\lambda([f]_\Lambda) = \int |f| d|\lambda|, \quad for\ [f]_\Lambda \in L^1(\Lambda),$$

or, alternately, the topology whose basis of neighborhoods of zero is the family of sets

$$\{[f]_\Lambda : [f]_\Lambda \in L^1(\Lambda), \sup_{i \in I} \int |f| d|\lambda_i| < \varepsilon\}$$

for every $\varepsilon > 0$ and every finite subset $\{\lambda_i\}_{i \in I} \subset \Lambda$. Let $\Gamma \subset ca(S)$ be a set for which the functional $[f]_\Lambda \mapsto \int f d\mu$ is well defined for every $\mu \in \Gamma$. The topology $\sigma(\Gamma)$ on $L^1(\Lambda)$ is determined by the family of semi-norms $\{q_\mu : \mu \in \Gamma\}$, where

$$q_\mu([f]_\Lambda) = |\int f d\mu|, \quad for\ [f]_\Lambda \in L^1(\Lambda).$$

We now present the corrected theorems.

Theorem 1. (Theorem III.2.1, [17]) For every $\tau(\Lambda)$-continuous linear functional φ on $L^1(\Lambda)$ there exists a measure $\mu \in ca(S)$, a finite subset $\{\lambda_i\}_{i\in I} \subset \Lambda$ and a constant $k \geq 0$ such that $|\mu| \leq k\sum_{i\in I}|\lambda_i|$ and

$$(1)\quad \varphi([f]_\Lambda) = \int f d\mu,\ for\ [f]_\Lambda \in L^1(\Lambda).$$

Conversely, if $\mu \in ca(S)$ is such that there is a finite subset $\{\lambda_i\}_{i\in I} \subset \Lambda$ and a constant $k \geq 0$ with $|\mu| \leq k\sum_{i\in I}|\lambda_i|$, then (1) defines a $\tau(\Lambda)$-continuous linear functional φ on $L^1(\Lambda)$.

The weak topology on $L^1(\Lambda)$ is the topology $\sigma(\Gamma)$, where Γ is the set of all measures $\mu \in ca(S)$ for which there is a finite subset $\{\lambda_i\}_{i\in I} \subset \Lambda$ and a constant $k \geq 0$ such that $|\mu| \leq k\sum_{i\in I}|\lambda_i|$.

Theorem 2. (Theorem III.5.1, [17]) Let $\Lambda \subset ca(S)$ and suppose Γ is the set of all measures $\mu \in ca(S)$ for which there exists a constant $k > 0$ and a finite subset $\{\lambda_i\}_{i\in I} \subset \Lambda$ such that $|\mu| \leq k\sum_{i\in I}|\lambda_i|$. The space $L^1(\Lambda)$ is $\tau(\Lambda)$-complete if and only if $L^1_{[0,1]}(\Lambda)$ is $\sigma(\Gamma)$-compact, that is, $L^1_{[0,1]}(\Lambda)$, the subspace of $L^1(\Lambda)$ defined in the following corollary, is weakly compact.

Corollary 3. (Corollary III.5.1, [17]) Suppose Ω is the set of all measures $\mu \in ca(S)$ for which there exists a finite subset $\{\lambda_i\}_{i\in I} \subset \Lambda$ with $\mu \ll \sum_{i\in I}|\lambda_i|$. Then $S(\Lambda)$ is $\tau(\Lambda)$-complete if and only if $L_{[0,1]}(\Lambda)$ is $\sigma(\Omega)$-compact, where

$$L^1_{[0,1]}(\Lambda) = \{f \in L^1(\Lambda) : 0 \leq f \leq 1\}.$$

There is another mistake in Kluvanek and Knowles [17] on p.67. Let $m : S \to X$ be a vector measure and suppose $\Lambda \subset ca(S)$ is a set of positive measures which is equivalent to m, that is, $p(m)(E) \to 0$ for every $p \in P$ if and only if $\lambda(E) \to 0$ for every $\lambda \in \Lambda$, where $p(m)$ is the p-semi-variation of m. Consider the following lemma:

Lemma 4. (Lemma IV.1.1, [17]) The integration mapping $m : L^1(m) \to X$ is continuous between the $\sigma(X'm)$ topology on $L^1(m)$ and the weak topology on X, where $X'm = \{< x', m >: x' \in X'\}$.

By this lemma, it is asserted in Kluvanek and Knowles [17] that $m : L^1(\Lambda) \to X$ is continuous between the $\sigma(\Omega)$ topology on $L^1(\Lambda)$ and the weak topology on X (Theorem IV.1.1, [17]) for the following reasons (i) $L^1(m) = L^1(\Lambda)$ as sets and (ii) $X'm \subset \Omega$, where Ω is the set of all measures $\mu \in ca(S)$ for which there exists a finite subset $\{\lambda_i\}_{i \in I} \subset \Lambda$ with $|\mu| \ll \sum_{i \in I} |\lambda_i|$. However, (i) is false. In general, $L^1(m)$ is only a subspace of $L^1(\Lambda)$. This can be seen by the following example.

Example 5. Let 2^N be the σ-field of all subsets of N, the set of positive integers. Define a vector measure $m : 2^N \to c_0$ by $m(A)(n) = \frac{1}{n} 1_A(n), A \subset N$. We also define a positive measure λ on S :

$$\lambda(A) = \sum_{n \in A} \frac{1}{n} a_n, A \subset N,$$

where (a_n) is a sequence of positive numbers with $\sum_{n=1}^{\infty} a_n \leq 1$. Hence $m(A) = 0$ if and only if $\lambda(A) = 0$ since they are all equivalent to $A = \emptyset$. Therefore $\Lambda = \{\lambda\}$ is equivalent to m.

Let $f : N \to R$ be a function defined by $f(n) = n$. Then $\int f d\lambda = \sum a_n$ and $(\int f dm)(n) = 1$ for every n. Therefore $f \in L^1(\lambda)$ but f does not belong to $L^1(m)$ since $\int f dm$ does not belong to c_0. Therefore $L^1(m) \neq L^1(\lambda)$.

It is clear now that, in general, $L^1(m) \subset L^1(\Lambda)$ but $L^1(m) \neq L^1(\Lambda)$. However, the subspace $L^1_{[0,1]}(m)$ of $L^1(m)$ and the subspace $L^1_{[0,1]}(\Lambda)$ of $L^1(\Lambda)$ are equal. As a result we can correct the theorem as follows.

Theorem 6. (Theorem IV.1.1,[17]) The integration mapping $m : L^1_{[0,1]}(\Lambda) \to X$ is continuous between the $\sigma(\Omega)$ topology on $L^1_{[0,1]}(\Lambda)$ and the weak topology on X.

Fortunately, this corrected theorem is sufficient to prove the following theorem.

Theorem 7. (Theorem IV.6.1,[17]) Let (T, S) be a measurable space, X be a quasi-complete locally convex space and $m : S \to X$ be a vector measure. Then the set $\overline{co}\ m(S)$ is weakly compact, where $\overline{co}\ m(S)$ is the closed convex hull of $m(S) = \{m(A) : A \in S\}$. If the vector measure is closed, then

$$\overline{co}\, m(S) = \{m(f) : [f]_\Lambda \in L^1_{[0,1]}(\Lambda)\}.$$

This theorem is important to the discussion of control systems, especially to the bang-bang control problem.

As a preparation for the discussion of control systems, let us define the concept of Liapunov vector measures.

Definition 8. A vector measure $m : S \to X$ is called a Liapunov vector measure if the set $m(S_E) = \{m(A) : A \in S_E\}$ is convex and weakly compact for every $E \in S$.

If X is quasi-complete, by Theorem 7, the condition in Definition 8 is equivalent to the requirement that $m(S_E)$ be convex and closed, that is, $\overline{co}\, m(S_E) = m(S_E)$ for $E \in S$.

Liapunov vector measures do not have to be closed. But we have criteria for a closed vector measure to be a Liapunov vector measure, which will be introduced later.

5.2 Control Systems

A control system is usually described by an ordinary or partial differential equation, but sometimes it can be expressed in terms of vector measures.

Three topics will be discussed in this chapter on control systems defined by vector measures: the bang-bang principle, time optimal control and controllability.

In the rest of this chapter, unless otherwise stated, we assume that X is a real, quasi-complete locally convex space. We also assume that (T, S) is a measurable space.

Definition 1. A sequence $m = (m_i)$ of closed vector measures $m_i : S \to X$, $i = 1, 2, \dots$, is called a control system (or a system in short) if $\sum_{i=1}^{\infty} x_i$ is convergent for any $x_i \in m_i(S)$, $i = 1, 2, \dots$.

Since $0 \in m_i(S)$, the convergence is unconditional.

There are two main reasons for choosing closed measures to define control systems. One is that m is closed whenever X is metrizable, which is the case for most of systems in practice. The other reason is that we have a richer theory on closed vector measures than on general vector measures. For instance, we will have criteria for a closed vector measure to be Liapunov.

Lemma 2.(Lemma IX.1.1, [17]) If $m = (m_i)$ is a control system and $f = (f_i)$ is a sequence of bounded measurable real functions on T, then $\sum_{i=1}^{\infty} m_i(f_i)$ converges.

Definition 3. A sequence $f = (f_i)$ of bounded S-measurable real functions on T is called a control. And the effect of a control $f = (f_i)$ on a control system $m = (m_i)$ is $m(f) = \sum_{i=1}^{\infty} m_i(f)$, that is, $\int f dm = \sum_{i=1}^{\infty} \int f dm_i$.

Usually there are restrictions on controls. Here we will use a set-valued function to express such restrictions.

Let CCR^{∞} be the family of all compact convex subsets of R^{∞}, the countable product of the real line treated as a locally convex space under the topology of co-ordinate-wise convergence. We call a set-valued function $F : T \to CCR^{\infty}$ measurable if for each $x' \in (R^{\infty})'$, the mapping

$$t \mapsto \sup\{< x', x >: x \in F(t)\}$$

from T to R is S-measurable. Such a function F is called bounded if there exists a compact set $K \subset R^{\infty}$ such that $F(t) \subset K$ for all $t \in T$.

Denote by $BM(R^{\infty}, S)$ the set of all $f = (f_i)$, sequences of bounded S-measurable real functions on T. For a set-valued function $F : T \to CCR^{\infty}$ and a control system

m we set

$$M_F(R^\infty, S) = \{f \in BM(R^\infty, S) : f(t) \in F(t) \; for \; all \; t \in T\}$$

and $A_F(m) = \{m(f) : f \in M_F(R^\infty, S)\}$.

Definition 4. The elements of $M_F(R^\infty, S)$ are called admissible controls and $A_F(m)$ is called the attainable set of control system m subject to the restriction F.

The definition of an attainable set is one of the most important definitions in control theory. Almost all control problems are related to attainable sets. Controllability, for example, if the attainable set of a system covers the whole space X, we say that the system has (global) controllability; and if the attainable set merely covers a neighborhood of a given point in X, we say that the system has local controllability at the point. We have the the following theorem on the structure of attainable sets.

Theorem 5.(Theorem IX.1.1, [17]) If $F : T \to CCR^\infty$ is bounded measurable, then the attainable set $A_F(m)$ for a control system m is convex, weakly compact subset of X.

Let us close this section with an example.

Example 6. We consider the temperature distribution $u(x, y, t; f)$ on the half plane $(x, y) \in R^+ \times R$ with control $f(y, t)$ on the boundary $x = 0$, where $R^+ = (0, \infty)$. If we assume that the initial temperature is zero, then $u = u(x, y, t; f)$ satisfies the diffusion equation:

$$(1) \;\; u_t = k\Delta u \;\; ,$$

$$u(x, y, 0; f) = 0, \;\; (x, y) \in R^+ \times R \;\; ,$$

$$u(0, y, t; f) = f(y, t), \;\; (y, t) \in R \times R^+,$$

where $k > 0$ is a constant and $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The solution of (1) can be expressed by

$$(2)\ \ u(x,y,t;f) = \frac{x}{8\pi k}\int_0^t\int_{-\infty}^{\infty}\frac{f(\eta,\tau)}{t-\tau}exp[-\frac{(y-\eta)^2+x^2}{4k(t-\tau)}]d\eta d\tau$$

for $(x,y,t) \in R^+ \times R \times R^+$.

Suppose the control problem is to choose a control f_0 such that the temperature along a line $y = y_0$ at a moment $t = t_0$ is prescribed as a continuous function $\phi(x)$ on R^+. Namely, we hope to choose $f = f_0$ to satisfy

$$(3)\ \ \phi(x) = \int\int_{[0,t_0]\times R} f(\eta,\tau)L(x,\tau)M(\eta,\tau)d\eta d\tau, \quad for\ \ x > 0,$$

where

$$L(x,\tau) = \frac{x}{8\pi k(t_0-\tau)}exp[-\frac{x^2}{4k(t_0-\tau)}]$$

and

$$M(\eta,\tau) = exp[-\frac{(y_0-\eta)^2}{4k(t_0-\tau)}]$$

for $(\tau,\eta) \in [0,t_0] \times R$.

If we choose a suitable measurable space and a suitable vector space, we can make (3) become an integral of f with respect to a vector measure. Indeed, we define $T = [0,t_0] \times R$, $S = B(T)$, the σ-algebra of Borel subsets of T and $X = C(R^+)$, the space of all continuous functions on R^+ with locally uniform convergence, and define $m : S \to X$ by

$$m(E)(x) = \int\int_E L(x,\tau)M(\eta,\tau)d\eta d\tau$$

for $x > 0$ and $E \in S$. Then m is a vector measure and $\phi = \int f dm$. Moreover, m is a closed vector measure since $C(R^+)$ is metrizable.

5.3 The Bang-Bang Principle

In 1946, Liapunov proved that the range of a non-atomic R^n-valued measure is convex and compact but this is not the case for X-valued measure when X is infinite

dimensional. As a consequence of this theorem, a very important feature of finite dimensional linear control systems, the bang-bang principle, is obtained. Namely, any point, which is reachable by a control taking values in a convex compact set U of R^n is reachable by a control taking values on the extreme points of U. Such controls are called bang-bang controls because in the case where $n = 1$ and $U = [0,1]$, controls taking extreme values 0 and 1 of U can be realized by switches jumping between off and on. Parallel to the fact that Liapunov theorem need not hold for infinite dimensional non-atomic vector measures, we do not expect the bang-bang principle for a infinite dimensional system in general. But we are interested in studying those preserving this feature.

For a set-valued function $F: T \to CCR^\infty$, we define $exF: T \to CCR^\infty$ by $(exF)(t) = exF(t)$, the set of extreme points of $F(t)$, for all $t \in T$.

Definition 1. Let $m = (m_i)$ be a control system and $F: T \to CCR^\infty$ a bounded measurable set-valued function. If $A_F(m_E) = A_{exF}(m_E)$ for every $E \in S$, then m is called F-Liapunov control system. In this case, we say that the system has the bang-bang principle and elements of $M_{exF}(R^\infty, S)$ are termed bang-bang controls.

We have a criterion for m to be F-Liapunov for every bounded measurable

$$F: T \to CCR^\infty.$$

Theorem 2.(Theorem IX.3.1, [17]) A control system $m = (m_i)$ is F-Liapunov for every bounded measurable $F: T \to CCR^\infty$ if and only if for every $u \in BM(R^\infty, S)$ not m-equivalent to 0, there exists a bounded measurable function $v: T \to R$ with $uv = (u_i v)$ not m-equivalent to 0 and $m(uv) = \sum_{i=1}^{\infty} m_i(u_i v) = 0$.

If F is chosen to be a special set-valued function $I: T \to CCR^\infty$ defined by $I(t) = \prod_{i=1}^{\infty} [0,1]$ for all $t \in T$, then we have a connection between the definition

of an F-Liapunov control system $m = (m_i)$ and the definition of a Liapunov vector measure (Definition 8 in Section 5.1).

Theorem 3.(Theorem 3.3, [18]) A control system $m = (m_i)$ is I-Liapunov if and only if each m_i is Liapunov, $i = 1, 2, \ldots$.

In particular, if m is reduced to a single component m_1, that is, $m_i = 0$ for $i \geq 2$, then the control system has the bang-bang principle if m_1 is Liapunov. In this case we often identify m with m_1 and call it a Liapunov control system.

There are many equivalent conditions for a closed vector measure to be Liapunov.

Theorem 4.(Theorem V.1.1, [17]) If $m : S \to X$ is a closed vector measure then the following properties are equivalent:

(i) m is a Liapunov vector measure;

(ii) for any set $E \in S$ not m-null there exists a function f in $BM(S)$ not m-null on E such that $m_E(f) = 0$;

(iii) for every function u in $BM(S)$ not m-null there exists a function $v \in BM(S)$ such that uv is not m-null but $m(uv) = 0$;

(iv) for every set $E \in S$ not m-null the integration mapping $m_E : L^\infty(m_E) \to X$ is not injective.

Applying (ii) in the theorem above to the vector measure m in Example 6 in Section 5.2, we know that m is Liapunov. Thus, if a temperature distribution ϕ is reachable by some control f taking values in $[0, 1]$, it can be reached by an f with the only two values 0 and 1.

5.4 Time Optimal Control

In this section time t will appear explicitly in the control systems. Let us specify the setting as follows: $T = \Omega \times [0, t_0]$ and $S = \Sigma \times B([0, t_0])$, where (Ω, Σ) is a measurable space, $t_0 > 0$, $B([0, t_0])$ is the Borel σ-algebra on $[0, t_0]$ and $\Sigma \times B([0, t_0])$

is the product σ-algebra of Σ and $B([0,t_0])$. For each $t \in [0,t_0]$, let $m(t) = m_{\Omega\times[0,t]}$ the restriction of m on $\Omega \times [0,t]$, which is a control system on (Ω, Σ), and $A(t) = A_F(m_t)$ with bounded measurable $F : \Omega \times [0,t_0] \to CCR^\infty$.

Definition 1. Given a fixed closed convex subset W of X, if there exists a $t^* \in [0,t_0]$ such that $A(t^*) \cap W \neq \emptyset$ and $t^* = \inf\{t \in [0,t_0] : A(t) \cap W \neq \emptyset\}$, then t^* is called the optimal time and the controls f^* satisfying $m(t^*, f^*) = m(t^*)(f^*) \in W$ are called time optimal controls.

We will need some hypotheses in our discussion:

(A) For some $t_1 \in [0,t_0]$, $A(t_1) \cap W \neq \emptyset$;

(B) For any $t_2 \in [0,t_0]$ and any $x' \in X'$,

$$\sup\{| < x', m(t)(f) - m(t_2)(f) > | : f \in BM_F(R^\infty)\} \to 0$$

as $t \searrow t_2$;

(C) For any control f, the function $t \mapsto m(t,f)$ from $[0,t_0]$ to X is continuous.

Theorem 2.(Theorem 3.1, [15]) Under hypotheses (A) and (B), an optimal time exists. If in addition (C) holds and $t^* > 0$, then time optimal controls $f^* = (f_i^*)$ satisfy the following necessary condition:

$$(1) \quad \sum_{i=1}^{\infty} \int_0^{t^*} \int_\Omega f_i^* d < x', (m_i)_{t^*} >$$
$$= \max\{\sum_{i=1}^{\infty} \int_0^{t^*} \int_\Omega f_i d < x', (m_i)_{t^*} >: f \in M_F(R^\infty)\}.$$

In particular, if $F(\omega,\tau) = \prod_{i=1}^{\infty}[-1,1]$ for all $(\omega,\tau) \in \Omega \times [0,t_0]$, then $|f_i| = 1$ $< x', (m_i)_{t*} > -a.e.$, $i = 1, 2, \ldots$.

Note that time optimal controls may not be unique in general. However, we can have the uniqueness under certain conditions.

Definition 3. For a fixed $t \in [0, t_0]$, if the vector measures $m_i(t)$, i=1,2,... , are defined by an integral with respect to the same scalar measure μ, then m(t) is called normal in X if for every $x' \in X'$, $x' \neq 0$, μ is $< x', m_i(t) >$-continuous.

Theorem 4.(Theorem 4.4, [15]) If $m(t)$ is normal in X for each $t \in [0, t_0]$, then the time optimal control is uniquely determined by (1) μ-a.e. and consequently bang-bang μ-a.e. .

In some special cases, we are able to have sufficient conditions for a control system $m(t)$ to be normal in X.

Suppose $\Omega \subset R^m$ and X is a space of real functions on $\overline{\Omega}$. Consider the control system $m(t)$ defined by

$$(2)\quad m(t)(E)(x) = \sum_{i=1}^{\infty} v_n(x) \int\int_E g_n(\omega, t, \tau) d\omega d\tau, E \in B(\Omega \times [0, t]),$$

where $B(\Omega \times [0, t])$ is the σ-algebra of Borel subsets of $\Omega_0 \times [0, t]$, Ω_0 is a subset of Ω and (v_n) is a sequence in X.

Lemma 5.(Lemma 5.6, [15]) The control system $m(t)$ is normal in X if the following conditions are satisfied.

(i) For almost all $\omega \in \Omega_0$, the functions $\tau \mapsto g_n(\omega, t, \tau)$ for all $\tau \in [0, t]$ are linear independent;

(ii) For any $x' \in X'$ and almost all $\omega \in \Omega_0$, the function $\tau \mapsto \sum_{n=1}^{\infty} < x', v_n > g_n(\omega, t, \tau)$, $\tau \in [0, t]$ is analytic.

(iii) The functions (v_n) span X.

As an example we consider the following parabolic boundary value problem:

$$(3)\quad \frac{\partial u}{\partial t}(x, t) - \frac{\partial^2 u}{\partial x^2}(x, t) + q(x) u(x, t) = 0, \quad (x, t) \in (0, 1) \times (0, t_0],$$

$$u(1, t) + \alpha \frac{\partial u}{\partial x}(1, t) = f(t),$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \ \ 0 < t < t_0; \quad u(x,0) = 0, \ \ 0 < x < 1,$$

where $t_0 > 0$ and $\alpha > 0$ are fixed parameters and $q(x) > 0$ is a fixed continuously differentiable function on [0,1].

For sufficiently smooth function f, the solution of (3) can be written as

$$(4) \ \ u(x,t) = \sum_{n=1}^{\infty} A_n \mu_n v_n(x) \int_0^t f(t) exp[-\mu_n(t-\tau)] d\tau,$$

where $A_n = \int_0^1 v_n(x)dx$ for $n \geq 1$, (v_n) and (μ_n) are, respectively, the normal eigenfunctions and eigenvalues of the boundary value problem

$$(5) \ \ -v''(x) + q(x)v(x) = \mu v(x), \quad x \in (0,1)$$

$$v'(0) = 0$$

$$v(1) + \alpha v'(1) = 0.$$

It is clear that the control system $m(t)$ defined by (4) is in the form of (2) and the analyticity and linear independence conditions of the lemma are satisfied, hence the system is normal in any space X spanned by the eigenfunctions (v_n). In particular, $C([0,1])$ is spanned by (v_n).

5.5 Controllability

In Section 5.3, the bang-bang principle says that " if a point is reachable by a control $f \in M_F(R^\infty, S)$, it can be reached by some $g \in M_{exF}(R^\infty, S)$." In Section 5.4, the first hypothesis is that the given subset W must be reachable, that is, "$A(t_1) \cap W \neq \emptyset$ for some $t_1 \in [0, t_0]$." In fact these are some conditions on controllability.

Definition 1. A control system is called (approximately) controllable if $A_F(m) = X (\overline{A_F(m)} = X)$.

We have seen that if F is bounded measurable, then $A_F(m)$ is weakly compact and therefore bounded. Hence it is impossible for m to be controllable or approximately controllable. However, we can discuss local controllability.

Definition 2. For a fixed point $z \in X$, if $z \in A_F(m)(z \in \overline{A_F(m)})$, we say that the control system is (approximately) controllable at z.

But even local exact controllability is hard to get. Egorov observed that, in infinite dimensional problems it is more realistic to try to approximate the desired point with a prescribed accuracy rather than to reach it exactly, it may not even be possible to reach it at all. (See Kluvanek and Knowles [18] for reference.) So let us focus on local approximate controllability.

Let us just consider a simple setting. Let $(\Omega, \Sigma, \lambda)$ be a finite measure space, $T = \Omega \times [0, t_0]$ for some $t_0 > 0$ and $S = \Sigma \times B([0, t_0])$, the product σ-algebra of Σ and $B([0, t_0])$ and let $\lambda \times l$ be the product measure on S. Suppose X is a Banach space and a control system $\{m\}$ is defined by a single vector measure $m : S \to X$. Given an element $z \in X$ and some $\varepsilon > 0$, we consider the problem of when $B(z, \varepsilon) \cap A_F(m) \neq \emptyset$, termed as "$B(z, \varepsilon)$ is reached," where $B(z, \varepsilon)$ is the closed ball of radius ε about z in X and $F(\omega, \tau) = [-1, 1]$ for all $(\omega, \tau) \in \Omega \times [0, t_0]$. For any $t \in [0, t_0]$, define $m(t) : L^\infty(\lambda \times l) \to X$ by $m(t)(f) = m(t, f) = \int_0^t \int_\Omega f dm(t)(\omega, \tau)$ for $f \in L^\infty(\lambda \times l)$. The symbol "$\| \, . \, \|$" stands for the norm in X (or in X') and "$\| \, . \, \|_p$ " for the norm in $L_p(\lambda \times l)(1 \leq p \leq \infty)$.

Then we have:

Theorem 3.(Theorem 6.7, 6.8, [15]) The set $B(z, \varepsilon)$ is reached in time t if and only if $| < x', z > | - \varepsilon \| x' \| \leq \| m(t)^*(x') \|_1 \quad for \ all \ x' \in X'$.

Corollary 4. A control system $\{m\}$ with $m : S \to X$ is controllable at z if and only if $| < x', z > | \leq \| m(t_0)^*(x') \|_1 \quad for \ all \ x' \in X'$.

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BIOGRAPHICAL SKETCH

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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